THE

NORMAL

ELEMENTARY GEOMETRY:

EMBRACING A BRIEF TREATISE ON

Mensuration and Trigonometry.

DESIGNED FOR

ACADEMIES, SEMINARIES, HIGH SCHOOLS, NORMAL SCHOOLS, AND ADVANCED CLASSES IN COMMON SCHOOLS.

BY

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PREFACE.

Progress in education is symbolized in the multiplication and improvement of text-books. They are showered upon us like the flowers of spring-time, until a new text-book is no longer a novelty. To-day the author sends forth this little volume on what it is hoped may be a mission of usefulness. It comes modestly claiming a welcome from the public as an addition to our educational literature, and in support of such claim the following statement of its object and peculiarities is presented.

Our text-books upon Geometry, though well adapted to our higher institutions, are, for a large class of schools, both too voluminous and difficult. In many of our Academies, Seminaries, High and Normal Schools, the time allotted to Geometry is too brief to allow the pupil to complete more than four or five books of the ordinary text-book: in consequence of which, all of that most important and practical part treating of the measurement of the surface and volume of prisms, pyramids, cylinders, cones, and spheres, must be omitted. To supply this defect and enable the pupil to acquire a fuller knowledge of the subject, the present volume has been written.

General Features.—In its adaptation to the class of pupils designated, this work is characterized by four general features. First, an abbreviation of the ordinary text-books; second, a simplification, so far as possible, of the methods of demonstration usually employed; third, examples to impart the power of making a practical application of the principles of the science; fourth, undemonstrated theorems, to cultivate the power of

original thought and investigation. These general characteristics will be briefly noticed.

ABBREVIATION.—In the abbreviation of the subject, the object has been to present the most valuable part of Geometry in about one-half of the space usually devoted to it. This object has been accomplished in two ways:—first, by an omission of all that is not essential to the final results; and secondly, by such a modification of the remainder as to preserve the chain of logic intact. The difficulty of this will be appreciated by those who remember that many propositions, apparently of little importance, are essential to the proof of others which follow them.

SIMPLIFICATION.—Much care has been taken to simplify the subject as far as possible. The author has endeavored to give the very simplest methods of treating special subjects, such as parallels, areas, volumes, the circle, etc.; and also the clearest and most concise methods of demonstrating individual theorems. The method of *infinites*, as applied to incommensurables, and the principle of *indivisibles*, in treating of volumes have contributed largely to this simplification and abbreviation. These methods are regarded by some as less satisfactory than the ordinary method of *reductio ad absurdum;* but it will be remembered that they have the authority of our most eminent mathematicians, and, being so much more simple and concise, are greatly to be preferred in such a work.

APPLICATIONS.—A radical defect of most of our text-books upon Geometry is that they present the subject so abstractly, that when the pupil has completed his course he is often unable to make any practical application of what he has learned. This defect has been supplied by the presentation of a collection of *practical examples* at the close of each book. With these, the pupil can see the application, the practical value of what he is doing, and will not only be able to make use of his knowledge, but will be incited to study the subject with more interest and earnestness.

THEOREMS FOR ORIGINAL THOUGHT.—Another general defect of

our text-books upon Geometry is the lack of matter for original thought, for the training of the inventive powers of the student. The pupil is required to learn the demonstrations of the text-book, but he has no undemonstrated theorems to test his own geometrical powers, and to train him in reasoning independently of the text-book. In view of this general defect, a collection of *theorems for original thought* has been given at the close of each book.

Geometry, in both the respects mentioned, has been treated quite differently from arithmetic and algebra. In these latter works, we have generally a large class of problems both for the application of the principles and the exercise of original thought. It is proper to remark, too, that several authors have realized this defect of Geometry, and have occasionally given some practical problems, and, in one or two instances, a collection of undemonstrated theorems. In the present work, such problems and theorems are an essential and prominent part of the plan.

SPECIAL FEATURES.—The attention of teachers is also respectfully invited to the following special features of the work:

1. The *Systematic Arrangement* of the subject-matter, it is thought, will be an acceptable feature of the work.

2. The *Analysis* at the beginning of each book is supposed to be valuable in giving the pupil a general idea of the object of the book, and thus often indicating the course of reasoning in its development.

3. The *Doctrine of Parallels*, resulting from the modern definition of an angle and parallel lines, now adopted by several authors, is here presented in its most simple and concise form.

4. The *Subject of Areas* in Book III., and *Volumes* in Book VI., are presented with great conciseness and simplicity by the use of the doctrine of infinites and indivisibles.

5. The *treatment of the circumference and area of a circle*, in their relation to π, is much more simple and logical than any thing the author has met. It is confidently believed that it will give pupils a clearer view of the subject than they usually acquire in the study of other text-books.

MENSURATION.—A formal treatise upon Mensuration is also appended, although the most of it is really presented in the practical exercises. A few rules are given without demonstration; such omissions may be supplied by the teacher.

TRIGONOMETRY.—The little treatise on Trigonometry presents the elements of the subject briefly, and contains about as much as the advanced pupils in our ordinary academies, seminaries, etc. should be required to learn. A short treatise on Analytical Trigonometry is appended for those who have time to study this very interesting and useful subject. The Trigonometry and Geometry will be bound together, and also separately, to accommodate schools of different grades.

A general acknowledgment of indebtedness is due to those who have previously written upon the subject of Geometry, especially to American and French authors, many of whose works have been examined with great interest and profit.

Thanking my friends for the generous appreciation bestowed upon my previous labors, I send forth this little volume, hoping that it may be as kindly welcomed, and that, in its mission of usefulness, it may aid in awakening a deeper interest in the beautiful science of *form*,—a science over which the ancient sages mused with such deep enthusiasm, and to which the achievements of modern art and invention are so largely indebted.

EDWARD BROOKS.

STATE NORMAL SCHOOL, Jan. 10, 1865.

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TRIGONOMETRY.

HISTORY

OF

GEOMETRY AND TRIGONOMETRY.

GEOMETRY is generally supposed to have had its origin in Egypt, where the annual overflowing of the Nile obliterated the landmarks and rendered it necessary to have recourse to mathematical measurement to re-establish them. This origin is indicated by the term itself, Geometry being from two Greek words, γη, *earth*, and μετρον, *measure*, signifying, literally, the measurement of the earth. But, whatever may have been the origin of the term, the natural tendency of the human mind to compare things in respect of their forms and magnitudes is so universal, that a geometry more or less perfect must have existed since the first dawn of civilization.

Geometry, originating in Egypt, is supposed to have been introduced into Greece by Thales, who lived about the year 650 B.C. Pythagoras, who lived about 570 B.C., was one of the earliest Greek geometers. He is supposed to have discovered the following principles:—1. Only three plane figures can fill up the space about a point; 2. The sum of the angles of a triangle equals two right angles; 3. The celebrated proposition of the square on the hypothenuse. Some say that in honor of this last discovery he sacrificed one hundred oxen. Plutarch says but one ox. Cicero doubts even that, as it was in opposition to his doctrines to offer bloody sacrifices, and suggests that they may have been images made of flour or clay.

The next geometer of eminence was Anaxagoras, who composed a treatise on the *quadrature of the circle*. Plato, the "poetical philosopher," delighted in the science, and cultivated it with great success, as is proved by his simple and elegant solution of the *duplication of the cube*. About fifty years after the time of Plato, Euclid collected the propositions which had been discovered by his predecessors, and formed of them his famous "*Elements*,"—a work of such eminent excellence that by many it is regarded, even at the present day, as the best text-book upon the subject of Elementary Geometry. It consists of fifteen books, thirteen of which are known to have been written by Euclid; but the fourteenth and fifteenth are supposed to have been added by Hypsicles of Alexandria.

Apollonius of Perga, about 250 years B.C., composed a treatise on *Conic*

Sections, in eight books. He is said to have given them their names, *parabola*, etc. About the same time flourished Archimedes, who distinguished himself in Geometry by the discovery of the beautiful relation between the sphere and cylinder. See Theorem XI. Book VII. He also distinguished himself by his work on *conoids* and *spheroids*, by his discovery of the exact *quadrature of the parabola*, and his very ingenious approximation to that of the circle.

Other geometers of eminence followed, among whom the most illustrious, perhaps, were Pappus and Diophantus; but the Greek Geometry, though it was afterward enriched by many new theorems, may be said to have reached its limits in the hands of Archimedes and Apollonius, and a long interval of seventeen centuries elapsed before this limit was passed. In 1637, Descartes published his Geometry, which contained the first systematic application of algebra to the solution of geometrical propositions. Soon after this followed the discovery of the infinitesimal calculus of Leibnitz and Newton; and from that time to the present Geometry has shared in the general progress of all mathematical sciences.

TRIGONOMETRY.—Trigonometry, it is generally believed, originated with the Greek astronomers of Alexandria. The solutions of the most useful cases of spherical triangles have been known from the time of Hipparchus, and the fundamental formulæ appear in the Analemma of Ptolemy.

The Greeks used the *chords* of the *double arcs*, instead of the *sines*. The *sines*, or *semi-chords*, were introduced by the Arabians, probably by the astronomer Albategnius. To the Arabs, who preserved and cultivated the sciences during the dark ages, this science is indebted also for several other improvements. Regiomontanus introduced the *tangents*, which did much to simplify the calculations.

The term *sine* seems to be derived from the Latin *sinus*, a bosom; the arc is supposed to represent a *bow*, and thus gets its name; the string, half of which represents the sine of half the arc, would come against the heart or bosom; hence the name *sine*. The terms *tangent* and *secant* are naturally derived from the old geometrical definitions. The *cosine* and *co-secant* of an arc mean the sine and secant of the complement, the *co* being merely an abbreviation of *complement*. They were first introduced by Gunter.

There are two methods of treating Trigonometry, known as the analytical and synthetic methods. The synthetic method regards the trigonometrical functions as lines, or geometrical magnitudes, and develops the science according to the laws of geometrical reasoning. The analytical method regards these functions as ratios or numbers, and develops the science by means of analytical formulas.

The modern or analytical method is superseding the ancient or geometrical method. This method is said to have been first introduced by Dr. Peacock. Professor De Morgan, however, one of the first English authorities, tells us that "Rheticus, who gave the first complete trigonometrical table, and invented the secant and co-secant to complete it, used the method of ratios."

LOGARITHMS.—Logarithms were invented by Lord Napier, Baron of Merchiston, in Scotland. His work upon them was first published in 1614; though it is probable that he had commenced the investigation of them as early as 1594. The invention is regarded as one of the most useful ever made. It gave the author so high a reputation that Kepler dedicated a work to him in 1617, and succeeding mathematicians have paid him the highest compliments.

Napier's system of logarithms was afterward improved by Henry Briggs, a contemporary of the inventor, and Professor of Geometry in Gresham College. Assuming 10 for the basis, he constructed a system of logarithms corresponding to our system of numeration, which is much more convenient for the ordinary purposes of calculation. The two systems are distinguished as the *Napierian* and *Briggean*, or the *Hyperbolic* and *Common* logarithms. The former are called *Hyperbolic* because they represent the area of a rectangular hyperbola between its asymptotes; the latter are called *Common* because they are those in common use.

Briggs calculated the logarithms to 14 places, with the index of all numbers between 1 and 20,000, and between 90,000 and 100,000, and published them in 1624. Adrian Vlacq, a native of Holland, computed the logarithms of numbers between 20,000 and 90,000, and thus completed what Briggs had begun: he reduced the tables, however, to 10 decimal places. Vlacq's treatise was published in 1628, and contained the logarithms of all numbers up to 100,000, and also the logarithms of the sines, tangents, and secants of every minute of the quadrant. In 1623, he published a work containing the logarithmic sines, cosines, tangents, and cotangents for every ten seconds of the quadrant, calculated from the natural sines, etc. of the *Opus Palatinum* of Rheticus.

In the same year the *Trigonometrica Britannica* was published at Gouda, which contained the logarithmic sines and tangents for the 100th part of every degree of the quadrant, together with a table of natural sines, tangents, and secants. These had been computed by Briggs. Since then, many different tables have been published. The most complete are those of Vlacq; but these are very scarce. *Hutton's Logarithms* and *Babbage's Logarithms of Numbers* are among the most accurate and convenient. For more information upon the subject, see *Brande's Encyclopedia,* from which most of this history is collated.

SUGGESTIONS TO TEACHERS.

THE author desires to present the following suggestions to those who may use this work:

1. Young pupils should have a preliminary drill upon concrete Geometry before taking up the text-book. Let them be required to cut out triangles, squares, etc. from paper, give their names, compare them, and draw them upon the board. In this manner a general idea of the subject, the figures treated of, and even the method of reasoning, may be obtained, and the transition from this to the abstract will be simple and easy.

2. In the recitation, the pupil should be required to construct his diagram upon the blackboard without the aid of the text-book, and then enunciate and demonstrate the theorem, care being taken that the language and reasoning be accurate. At the close of the demonstration, those of the class who have noticed errors, upon being called upon by the teacher, should rise and point them out; after which the teacher may make any criticisms or explanations he may think proper.

3. With quite young pupils, and those whose time for the study is limited, the *theorems for original thought* may be omitted; with others, however, these exercises will be found to be of great value. They can be given in connection with the demonstrations of the book, or lessons may be assigned upon them after completing the book to which they belong, or they may be omitted until review. The latter method will be generally preferred.

The *Practical Exercises* should be solved by all classes. The easier problems may be assigned in connection with the theorems which they illustrate; the others may be deferred until the book upon which they depend is completed. The most difficult problems may be omitted until the whole Geometry is completed.

ELEMENTARY GEOMETRY.

INTRODUCTION.

LESSON I.

SUBJECT-MATTER OF GEOMETRY.

EVERY object that we can see occupies some portion of space, and has extent and form. If we consider some object, as this book, for instance, we will perceive that it has length, breadth, and thickness. These are called the dimensions of the book.

If, now, we remove the book from before us, we can still imagine the space which it filled to be in the form of a book. This space, of course, will not be a material thing like the book, but it will have form and extent the same as the book had. Such definite portions of space, their forms and extent, are the things considered in Geometry.

These limited portions of space are called *Volumes*. A volume has length, breadth, and thickness, and these are called its dimensions. We should be careful to distinguish the geometrical volume, which is a portion of space, from the solid body, which occupies space. The one is material, the other is immaterial; the one is *real body*, the other is *ideal body* or *pure form*. It is ideal body or pure form that is treated of in Geometry.

2

Let us now consider this ideal body or volume a little more closely. First, we will notice that it is distinctly separated from the surrounding space. That which so separates it is called a *Surface;* and, since that which bounds the volume forms no part of the volume, it will be seen that a surface has no thickness, and possesses, therefore, but two dimensions,—length and breadth.

If we consider one of these bounding surfaces, we will see that it also is limited or bounded. That which limits a surface is called a *Line;* and since that which limits a surface forms no part of the surface, it is seen that a line has only one dimension,—length.

Again, if we examine one of these lines, we will see that its ends are limited. This limit is called a *Point;* and, since the limit forms no part of the line, a point has neither length, breadth, nor thickness, but position only.

Now, although we have considered a point as the limit of a line, a line as the limit of a surface, and a surface as the limit of a volume, yet each of these may be regarded in a purely abstract manner, distinct from each other. Thus, we may consider points without regard to lines, lines without reference to surfaces, and surfaces without reference to volumes.

We have now attained a conception of the ideas of Geometry by passing from a body to an abstract volume, from this volume to a surface, from a surface to a line, and from a line to a point. This is the method of analysis, and is, without doubt, the method in which these ideas were primarily attained. They may, however, also be attained by synthesis, in the following manner.

Fix upon the idea of a point in space. Now, suppose

this point to move, and we have a line; suppose the line to move in a particular manner, and we have a surface; suppose the surface to move, and we have a volume.

These lines, surfaces, and volumes, of which we have attained the idea, are the fundamental quantities of Geometry. A quantity, you will remember, is any thing that can be measured. When one line crosses another, their divergence may be measured: hence we have a fourth kind of geometrical quantity, called *angles*. We are now prepared to define Geometry.

Geometry is that science which treats of the properties and relations of geometrical magnitudes. Its subject-matter are lines, surfaces, volumes, and angles.

LESSON II.

REASONING OF GEOMETRY.

THE subject-matter of Geometry, we have seen, are lines, surfaces, volumes, and angles. These general conceptions give rise to many special forms; these special forms are described, and such descriptions constitute the *Definitions* of the science.

When we consider these special forms of quantity, as well as quantity in general, we perceive some truths concerning them that are self-evident,—that must be true, since they cannot be conceived as untrue. These self-evident truths are called *Axioms*.

The science of Geometry begins with these primary ideas of space and the self-evident truths arising out of them, and from these, as a basis, rises to the higher truths by a process of reasoning. The axioms and definitions are,

therefore, said to be the basis of the science of Geometry. The definitions present the subjects upon which we reason; the axioms give the laws which guide us in the reasoning process. From these we trace our way, step by step, to the loftiest and most beautiful truths of the science, by the simple process of comparison. This process of comparison is called *reasoning;* and to this we now call attention.

REASONING.—All Reasoning is *comparison.* A comparison requires a standard or basis, and this standard is the *simple*, the *axiomatic*, the *known.* To these we bring the *complex*, the *theoretic*, the *unknown*, and learn to understand them by comparing the *complex* with the *simple*, the *theoretic* with the *axiomatic*, the *unknown* with the *known.*

There are two distinct methods of geometrical reasoning, which may be distinguished as the *analytic* and *synthetic* methods. The analytic method is adapted to the discovery of truth; the synthetic method, to the proving of a truth when it has already been discovered.

SYNTHETIC METHOD.—The synthetic method, which is generally employed in proving a truth which is already known, is called *demonstration.* There are two distinct methods of demonstration, called the *Direct* and the *Indirect Method.*

The simplest form of the *Direct Method* is that in which figures are directly compared by applying one to another. This is called the method by *superposition.* The more general form of the direct method is that in which truths are proven by a reference to the definitions and axioms, or some principle previously proven.

The *Indirect Method,* known as the *reductio ad absurdum,* consists in supposing the proposition to be proven not to

be true, and then showing that such an hypothesis leads to a contradiction of some known truth. This is frequently used to prove the converse of a proposition, when there is no good direct method; it is also used in incommensurable quantities.

There are two errors of demonstration into which young pupils are liable to fall. The first is called *Reasoning in a Circle;* the second is *Begging the Question.* We reason in a circle when, in demonstrating a truth, we employ a second truth which cannot be proven without the aid of the first. We are said to beg the question when, in order to establish a proposition, we employ the proposition itself.

ANALYTIC METHOD.—The analytic method begins with the thing required, and by tracing the relations of the various parts we arrive at some known truth. It is a kind of going back from the result sought by a chain of relations to what has been previously established. In a demonstration, we pass through every step from the simplest self-evident truth to the highest deductions of the science; in the process of analysis, we pass over every step from the latter truths down to the simplest.

Analysis is the method of discovery; synthesis, of demonstration. The one has for its object to find unknown truths; the other, to prove known ones. Frequently both methods are employed simultaneously, when the object is to discover new relations, or the solution of new problems; but when we wish to prove to others the truths we have discovered, the synthetical method is usually preferred.

LESSON III.

GEOMETRICAL LANGUAGE.

LANGUAGE is the instrument of thought and the medium of expression. All thinking is by means of language; and the more concise and perfect the language, the more profound and searching is our thought. The language of mathematics differs somewhat from that of ordinary usage, in being more concise and more definite in its use.

Much of the language of mathematics is symbolical; that is, a symbol is used in place of the written word. There are three classes of symbols in Geometry: *symbols of quantity*, *symbols of operation*, and *symbols of relation*.

The SYMBOLS OF QUANTITY are usually pictured representations of the quantities considered. Sometimes, however, the letters of the alphabet are used to indicate them.

The SYMBOLS OF OPERATION are as follow:—

The *Sign of Addition*, $+$, called *plus;* thus, $A + B$, denotes that B is to be added to A.

The *Sign of Subtraction*, $-$, called *minus;* thus, $A - B$, denotes that B is to be subtracted from A.

The *Sign of Multiplication*, $\times$; thus, $A \times B$, denotes that A is to be multiplied by B.

The *Sign of Division*, $\div$; thus, $A \div B$, denotes that A is to be divided by B.

The *Exponential Sign;* thus, A^4, denotes that A is used four times as a factor, or is raised to the fourth power.

The *Radical Sign*, $\sqrt{}$; thus, $\sqrt{A}$, $\sqrt[3]{B}$, denotes that the square root of A and the cube root of B are to be extracted.

The *Parenthesis* and *Vinculum* denote that the quantity

is to be operated upon as a whole; thus, $(A+B)\times C$, or $\overline{A+B}\times C$, denotes that the sum of A and B is to be multiplied by C.

The Symbols of Relation are as follow:—

The *Sign of Equality*, $=$; thus, $A=B+C$, denotes that A is equal to the sum of B and C.

The expression of the equality of two quantities is an equation; thus, $A=B+C$, is an equation. The part on the left of the sign of equality is the *first member;* that on the right is the *second member.*

The *Sign of Inequality*, $>$ or $<$; thus, $A>B$, denotes that A is greater than B. The greater quantity is at the opening of the sign.

The *Sign of Ratio*, $:$; thus, $A:B$, denotes the ratio of A to B.

The *Sign of Equal Ratios*, $::$; thus, $A:B::C:D$, denotes that the ratio of A to B *equals* the ratio of C to D.

We present also a few combinations of these symbols, called *formulas*, which will be found valuable in some of the demonstrations.

1. $A\times B+C\times B=(A+C)\times B.$
2. $\frac{1}{2}A\times B-\frac{1}{2}B\times C=\frac{1}{2}(A-C)\times B.$
3. $(A+B)^2=A^2+2A\times B+B^2.$
4. $(A-B)^2=A^2-2A\times B+B^2.$
5. $(A+B)\times(A-B)=A^2-B^2.$

DEFINITION OF TERMS.

An Axiom is a self-evident truth.

A Theorem is a truth to be demonstrated.

A Problem is a question to be solved.

A Postulate is a problem whose solution is self-evident.

A COROLLARY is an obvious consequence of one or more propositions.

A SCHOLIUM is a remark upon one or more propositions.

Theorems, Axioms, Problems, and Postulates, are all called *Propositions*.

An HYPOTHESIS is a supposition made in the statement of a proposition, or in its demonstration.

NOTE.—In making references, A. stands for Axiom; B. for Book; C. for Corollary; D. for Definition; I. for Introduction; Th. for Theorem; P. for Problem; S. for Scholium. In referring to another Book, the number of the book is given; in referring to the same Book, the number of the Book is not given.

ELEMENTARY GEOMETRY.

BOOK I.

DEFINITIONS.

1 GEOMETRY is the science which treats of the properties and relations of geometrical magnitudes.

2. A GEOMETRICAL MAGNITUDE is some definite element of space. It is a line, a surface, a volume, or an angle.

3. A POINT is that which has position, but no magnitude.

4. A LINE is that which has length, but no breadth or thickness. Lines are *straight* or *curved.*

5. A STRAIGHT LINE is one which has the same direction at every point: as, *AB*.

6. A CURVED LINE is one which changes its direction at every point: as, *CD*.

The word *line* used alone, means a *straight line;* the word *curve,* alone, means a *curve line.*

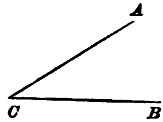
7. A SURFACE is that which has length and breadth, without thickness. Surfaces are *plane* or *curved.*

8. A PLANE is a surface such that if any two of its points be joined by a straight line, every part of that line will lie in the surface.

9. A Volume is that which has length, breadth, and thickness.

PLANE ANGLES.

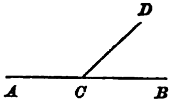
10. An Angle is the difference of direction, or the divergence, of two lines proceeding from a common point.



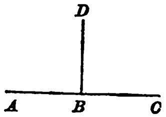
The point from which the lines proceed is called the *vertex* of the angle; the lines themselves are the *sides* of the angle.

An angle is named by the letter at the vertex, or by the three letters with the letter at the vertex in the middle. Thus, we say the angle *C*, or the angle *ACB*.

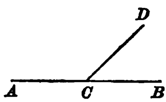
11. Adjacent Angles are those formed on opposite sides of a straight line which meets another straight line; thus, *ACD* and *BCD* are adjacent angles.



12. A Right Angle is an angle formed by one straight line meeting another, making the adjacent angles equal. The first line is then said to be *perpendicular* to the other.



13. An Obtuse Angle is one which is greater than a right angle; as, *ACD*.

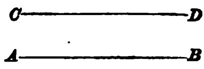


An Acute Angle is one which is less than a right angle; as, *DCB*.

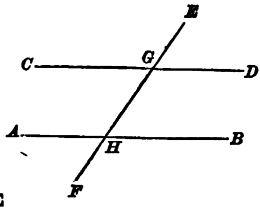
Obtuse and acute angles are called *oblique* angles, in distinction from right angles.

PARALLEL LINES.

14. Parallel Lines are those which have the same direction; as, *AB* and *CD*.



When a straight line intersects two parallel straight lines, the angles formed take particular names. Suppose the line EF to intersect the parallels AB and CD; then—



1. INTERIOR ANGLES ON THE SAME SIDE are those which lie within the parallels, on the same side of the *secant*, or intersecting line; thus, CGH and AHG; also, HGD and GHB;

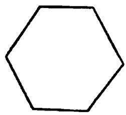
2. ALTERNATE INTERIOR ANGLES lie within the parallels, on different sides of the secant line, but not adjacent; as, CGH and GHB;

3. OPPOSITE EXTERIOR AND INTERIOR ANGLES lie on the same side of the secant line, one without and the other within the parallels, but not adjacent; as, EGD and GHB.

PLANE FIGURES.

15. A PLANE FIGURE is a plane bounded by lines either straight or curved.

16. A POLYGON is a plane figure bounded by straight lines. These lines are called *sides* of the polygon; taken together, they form the *perimeter* of the polygon.



17. A POLYGON of three sides is called a *triangle;* of four sides, a *quadrilateral;* of five sides, a *pentagon;* of six sides, a *hexagon;* of seven sides, a *heptagon;* of eight sides, an *octagon;* of nine sides, a *nonagon;* of ten sides, a *decagon*, etc.

18. An EQUILATERAL POLYGON is one whose sides are equal. An *Equiangular Polygon* is one whose angles are equal.

Two polygons are *mutually equilateral* when their sides

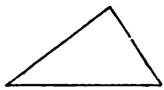
are respectively equal. Two polygons are *mutually equiangular* when their angles are respectively equal.

19. A DIAGONAL of a polygon is a line joining the vertices of two angles, not consecutive.

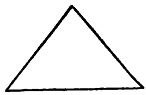
TRIANGLES.

20. A TRIANGLE is a polygon of three sides and three angles. Triangles are classified by their sides and their angles.

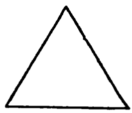
21. A SCALENE TRIANGLE is one in which the three sides are unequal.



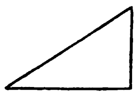
22. An ISOSCELES TRIANGLE is one which has two of its sides equal.



23. An EQUILATERAL TRIANGLE is one which has its three sides equal.



24. A RIGHT-ANGLED TRIANGLE is one which has one right angle. The side opposite the right angle is called the *hypothenuse*.



25. An ACUTE-ANGLED TRIANGLE is one in which all the angles are acute.

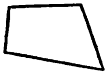
26. An OBTUSE-ANGLED TRIANGLE is one which has one obtuse angle.

Triangles are the simplest of all polygons, since three sides are the least number that can bound a plane figure. The properties of polygons are determined by analyzing them into triangles.

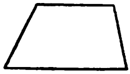
QUADRILATERALS.

27. A Quadrilateral is a polygon of four sides and four angles. There are three classes:—

1. The Trapezium is a quadrilateral having no two sides parallel.



2. The Trapezoid is a quadrilateral having two of its opposite sides parallel.

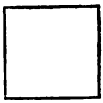


3. The Parallelogram is a quadrilateral having its opposite sides parallel.

28. Parallelograms are divided, from their angles, into two classes,—right-angled and oblique-angled parallelograms.

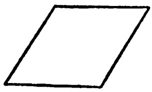
1. A Rectangle is a parallelogram whose angles are right angles.

A Square is an equilateral rectangle.



2. A Rhomboid is a parallelogram whose angles are oblique.

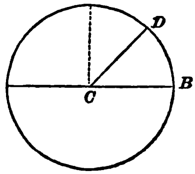
A Rhombus is an equilateral rhomboid.



THE CIRCLE.

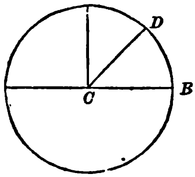
29. A Circle is a plane figure bounded by a curve line, every point of which is equally distant from a point within, called the *centre*.

The CIRCUMFERENCE is the bounding line; any part of the circumference is called an *arc*. A line through the centre having its ends in the circumference is a *diameter;* a line from the centre to the circumference is the *radius.*



30. The circumference of a circle is used to measure angles. An angle having its vertex at the centre is measured by the arc included between its sides; thus, the arc *BD* measures the angle *DCB.*

To measure angles, the circumference is divided into 360 equal parts, called *degrees.* Each degree is divided into 60 equal parts, called *minutes;* each minute into 60 equal parts, called *seconds.* Degrees, minutes, and seconds are marked thus, $^\circ$ $'$ $''$; $16^\circ\ 24'\ 32''$ are read, 16 degrees, 24 minutes, and 32 seconds.



A right angle, it will be seen, is measured by 90°; half a right angle, by 45°; two right angles, by 180°; four right angles, by 360°.

AXIOMS.

31. An AXIOM is a self-evident truth. There are two classes of axioms in Geometry. First, those which pertain to quantity in general; second, those which arise out of the special forms of geometrical quantity.

FIRST CLASS.

1. Things which are equal to the same thing are equal to each other.
2. If equals be added to equals, the sums will be equal.

3. If equals be subtracted from equals, the remainders will be equal.

4. If equals be added to or subtracted from unequals, the results will be unequal.

5. If equals be multiplied by equals, the products will be equal.

6. If equals be divided by equals, the quotients will be equal.

7. The whole is greater than any of its parts.

8. The whole equals the sum of all its parts.

SECOND CLASS.

9. Only one straight line can be drawn connecting two given points.

10. A straight line is the shortest distance from one point to another.

11. All right angles are equal to each other.

12. Parallel straight lines cannot meet each other when produced.

13. Through a given point only one straight line can be drawn parallel to a given line.

Corollary. From axiom 10, it is evident that either side of a triangle is less than the sum of the other two sides.

POSTULATES.

32. The following postulates are self-evident problems resulting from the preceding definitions:—

1. A straight line can be drawn from one point to another.

2. A straight line may be prolonged to any length.

3. A line or an angle may be bisected.

4. An angle may be described equal to a given angle.

5. A line may be drawn through a given point parallel to a given line.

6. A perpendicular may be drawn to a given line from a point on the line or without it.

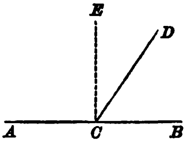
Analysis of Book I.—Book I. treats mainly of angles, parallel lines, triangles, and parallelograms. It treats of the angles formed by one line meeting or cutting another, of the angles formed by one line cutting two parallel lines, of the equality and inequality of triangles, of the sum of the angles of a triangle, of the relation of the angles and sides of a parallelogram, and of the exterior and interior angles of a polygon. It is thus seen that the idea of angles is a prominent, if not the principal one of the book.

OF ANGLES.

THEOREM I.

When one straight line meets another straight line, the sum of the two adjacent angles equals two right angles.

Let the straight line CD meet the straight line AB at the point C; then will $ACD + DCB =$ two right angles.



For, at the point C, draw EC perpendicular to AB; then (D. 12,) the angles ACE and ECB are both right angles. Now,

$ACD = a$ *right angle* $+ ECD$; and

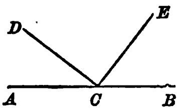
$DCB = a$ *right angle* $- ECD$; hence, adding,

we have, $ACD + DCB =$ *two right angles.*

Therefore, when a straight line meets another straight line, the sum of the two adjacent angles equals two right angles.

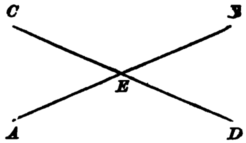
Cor. 1. If one of the angles ACD or DCB is a right angle, the other is also a right angle.

Cor. 2. The sum of all the angles formed on the same side of a straight line by drawing lines to any point of that line, is equal to two right angles. For their sum is equal to the sum of ACD and DCB, which is equal to two right angles, according to the proposition.



THEOREM II.

When two straight lines intersect each other, the opposite or vertical angles are equal.



Let the two straight lines AB and CD intersect each other at the point E; then will AEC be equal to BED.

For, since CE meets AB, the angle $AEC + CEB =$ two right angles (Th. I.); and since BE meets CD, the angle $CEB + BED =$ two right angles; but things which are equal to the same thing are equal to each other (A. 1); hence,

$$AEC + CEB = CEB + BED.$$

Taking from each sum the common angle CEB, there remains (A. 3),

$$AEC = BED.$$

In a similar manner it may be shown that the angle AED equals CEB. Therefore, etc.

Cor. 1. The sum of the four angles formed by the intersection of two lines is equal to four right angles.

Cor. 2. The sum of all the angles that can be formed about a point is equal to four right angles.

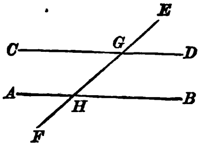
PARALLEL LINES.

THEOREM III.

If a line intersect two parallel lines;

1. *The opposite exterior and interior angles will be equal.*
2. *The alternate angles will be equal.*
3. *The sum of the interior angles on the same side will be equal to two right angles.*

Let the line EF intersect the two parallels AB and CD; then,



First. The angle EGD is equal to GHB. For, since HB and GD are parallel, they have the same direction; hence, they must diverge equally from the line EF; therefore, the difference of direction or divergence of GE and GD must be equal to the divergence of HE and HB, or the angle EGD equal to GHB. In the same way it may be shown that $FHB = HGD$.

Second. The two alternate angles CGH and EHB will be equal. For, CGH equals EGD (Th. II.); but EGD equals EHB; therefore, $CGH = GHB$ (A. 1); and in the same manner it may be shown that AHG equals HGD.

Third. The sum of the two interior angles GHB and HGD equals two right angles. For, $EGD + DGH =$ two right angles (Th. I.); but $EGD = EHB$; hence, $GHB + HGD =$ two right angles. In the same way it may be shown that $AHG + CGH$ equals two right angles. Therefore, etc.

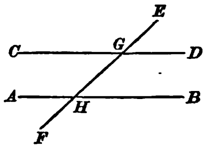
Cor. If a line is perpendicular to one of two parallels, it is perpendicular to the other also. For, if EGD were a right angle, its equal EHB would be a right angle also.

THEOREM IV.

Conversely.—*If a straight line meets two other straight lines, these two lines will be parallel;*

1. *When the opposite exterior and interior angles are equal.*
2. *When the alternate angles are equal.*
3. *When the sum of the two interior angles on the same side is equal to two right angles.*

Let the straight line EF meet the two straight lines AB and CD; then,



First. If the angles EGD and EHB are equal, the lines are parallel. For, since EGD and EHB are equal, the lines GD and HB must diverge equally from EF; hence, they have the same direction, and are, therefore, parallel (D. 14).

Second. If the alternate angles CGH and GHB are equal, the lines are parallel. For, since CGH equals EGD (Th. II.), when CGH equals GHB, EGD equals GHB; but then the lines are parallel, as has just been shown; hence, the lines are parallel when the alternate angles CGH and GHB are equal.

Third. If the sum of the two interior angles GHB and HGD equals two right angles, the lines are parallel. For, $EGD + HGD =$ two right angles (Th. I.); hence, $EGD + HGD = HGD + GHB$ (A. 1). Taking HGD from each, we have $EGD = GHB$; but then the lines are parallel, according to the first part of the theorem; hence, they are parallel when $GHB + HGD$ equals two right angles. Therefore, etc.

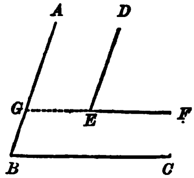
Cor. If two lines are perpendicular to the same line, they are parallel. For, if EGD and GHB are both right angles, the lines CD and AB are parallel, since the sum of the angles equals two right angles.

THEOREM V.

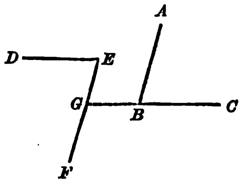
If two angles have their sides parallel and both lying in the same direction or in opposite directions, they are equal.

First. Let the angles ABC and DEF have their sides parallel and lying in the same direction; then will ABC equal DEF. For, prolong FE to G. Then, since AB and

DE are parallel, *DEF* equals *AGE* (Th. III.); and since *GF* and *BC* are parallel, *AGE* equals *ABC* (Th. III.); hence, *DEF* equals *ABC* (A. 1).



Second. Let the angles *ABC* and *DEF* have their sides parallel and lying in opposite directions; then will *ABC* equal *DEF*. For, prolong *CB* to *G*. Then *ABC* equals *EGB* (Th. III.); but *EGB* equals *DEF*, being alternate; hence, *ABC* equals *DEF*. Therefore, etc.

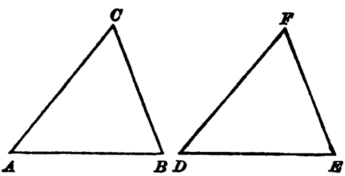


TRIANGLES.

THEOREM VI.

If two triangles have two sides and the included angle of the one equal to two sides and the included angle of the other, each to each, the triangles will be equal in all their parts.

Let the triangles *ABC* and *DEF* have the side *AB* equal to *DE*, *AC* to *DF*, and the angle *A* equal to the angle *D*; then will the triangle *ABC* be equal to the triangle *DEF*.

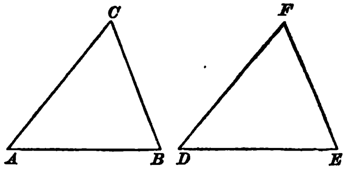


For, apply the triangle *ABC* to the triangle *DEF*, placing the side *AB* upon the equal side *DE*; then, since the angle *A* equals the angle *D*, the side *AC* will take the direction of *DF*, and the point *C* will coincide with *F*, since the two lines are equal; and the side *CB* will coincide with the side

FE (A. 9). Therefore, the triangles coincide and are equal in all their parts. Therefore, etc.

THEOREM VII.

If two triangles have two angles and the included side of the one equal to two angles and the included side of the other, each to each, the triangles will be equal in all their parts.



Let ABC and DEF be two triangles having the angle A equal to the angle D, the angle B equal to the angle E, and the included side AB equal to the included side DE; then will the two triangles be equal in all their parts.

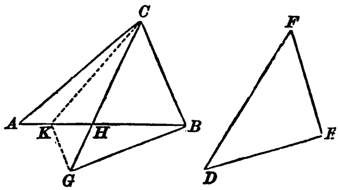
For, apply the triangle ABC to the triangle DEF, placing the side AB upon DE, the point A upon D, and the point B upon E; then, since the angle A equals the angle D, the side AC will take the direction DF, and the point C will be found somewhere in the line DF; and since the angle B equals the angle E, the side BC will take the direction EF, and the point C will be found somewhere in EF. Hence, the point C being in the two lines DF and EF, must be at their intersection; consequently, the triangles coincide and are equal in all their parts. Therefore, etc.

THEOREM VIII.

If two triangles have two sides of the one equal to two sides of the other, each to each, and the included angles unequal, the third side will be greater in the triangle having the greater included angle.

Let ABC and DEF be two triangles in which $AC = DF$,

$BC = EF$ and $ACB > DFE$; then will AB be greater than DE.

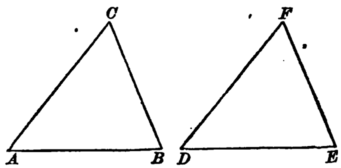


For, at the point C make the angle $BCG = EFD$, make $CG = FD$, and draw BG; then will the triangle CGB equal DFE and GB equal DE (Th. VI.). Draw CK, bisecting the angle ACH, and draw also GK; the two triangles ACK and KCG are equal (Th. VI.), and $AK = KG$. Now, $KG + KB > GB$; hence $AK + KB$, or AB, is greater than GB or its equal DE.

The same demonstration will apply when the point G falls within AB. If it falls upon AB, the theorem is true by A. 7. Therefore, *if two triangles*, etc.

THEOREM IX.

If two triangles have the three sides of the one equal to the three sides of the other, each to each, the triangles will be equal in all their parts.



Let ABC and DEF be two triangles, having AB equal to DE, AC to DF, and BC to EF; then will the triangles be equal in all their parts.

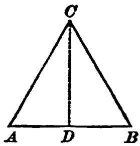
For, since AC and AB are respectively equal to DF and DE, if the angle A were greater than D, BC would be greater than EF (Th. VIII.); and if A were less than D, BC would be less than EF, for the same reason. But BC is equal to EF, therefore the angle A must be equal to D.

In the same way it may be shown that the angle C equals F, and the angle B equals E. Therefore, etc.

THEOREM X.

In an isosceles triangle the angles opposite the equal sides are equal.

Let ABC be an isosceles triangle, having the side AC equal to the side BC; then will the angle A be equal to the angle B.



Join the vertex C and the middle point of the base AB; then in the two triangles ADC and CDB, AC equals BC, DC is common, and AD equals DB; hence, the two triangles are equal in all their parts (Th. IX.), and the angle A is equal to the angle B. Therefore, etc.

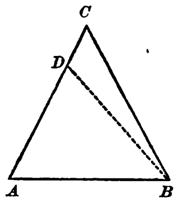
Cor. 1. *The line CD bisects the angle ACB*, since ACD equals DCB; *and is also perpendicular to the base*, since the angles ADC and CDB are equal, and are, therefore, right angles (D. 12).

Cor. 2. Hence, also, an equilateral triangle is equiangular; that is, it has all its angles equal.

THEOREM XI.

Conversely.—*If two angles of a triangle are equal, the sides opposite them are also equal, and the triangle is isosceles.*

Let ABC be a triangle, having the angle A equal to the angle B; then will the side AC be equal to BC.

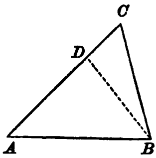


For, if AC and CB are not equal, suppose one of them, as AC, to be the greater. Then, take AD equal to BC, and draw DB. Now, in the triangles ABC and ABD, we have the side AD

equal to BC, by construction, the side AB common, and the included angle ABC equal to the included angle DAB, by hypothesis; hence, the two triangles ABD and ABC are equal (Th. VI.); that is, a part equal to the whole, which is impossible (A. 7). Hence, the hypothesis that AC and BC are unequal is false; therefore, they are equal, and the triangle is isosceles. Therefore, etc.

THEOREM XII.

In any triangle the greater side is opposite the greater angle, and, conversely, the greater angle is opposite the greater side.



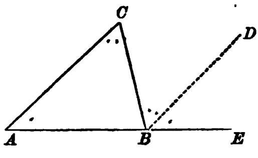
In the triangle ABC, let the angle ABC be greater than CAB; then will AC be greater than BC.

For, draw BD, making the angle $ABD = DAB$; then will $AD = DB$ (Th. XI.). To each add DC and we have $AD + DC = DB + DC$; but $DB + DC > BC$ (A. 10); hence, $AD + DC$, or AC, is greater than BC.

Conversely. Let the side $AC > BC$; then will the angle $ABC > CAB$. For, if $ABC < CAB$, $AC < BC$, from what has just been proved; and if $ABC = CAB$, $AC = BC$ (Th. XI.); but both of these results are contrary to the hypothesis; hence, ABC must be greater than CAB. Therefore, etc.

THEOREM XIII.

In every triangle the sum of the three angles is equal to two right angles.



Let ABC be a triangle; then will the sum of its three angles, A, B, C, be equal to two right angles.

For, prolong AB, and draw BD

parallel to AC; then, since the parallels AC and BD are cut by AE, the angle A is equal to the opposite exterior angle DBE (Th. III.). In like manner, since the parallels are cut by BC, the alternate angles C and CBD are equal; hence, the sum of the three angles of the triangle is equal to the sum of the angles ABC, CBD, DBE; but this latter sum equals two right angles (Th. I. C. 2); therefore, the sum of the three angles of the triangle equals two right angles. Therefore, etc.

Cor. 1. If two angles of a triangle are given, the third will be found by subtracting their sum from two right angles, or 180°.

Cor. 2. A triangle cannot have more than one right angle; for if there were two the third angle would be zero.

Cor. 3. A triangle can have only one obtuse angle, but must have at least two acute angles.

Cor. 4. In a right-angled triangle the sum of the two acute angles equals one right angle, or 90°.

Cor. 5. In every triangle ABC, the exterior angle CBE is equal to the sum of the two interior opposite angles A and C.

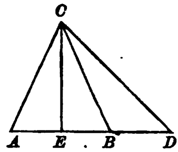
Scholium. This theorem may be demonstrated by drawing a line parallel to either of the other sides of the triangle. Let the pupils be required to do it.

THEOREM XIV.

If from a point without a straight line a perpendicular be let fall on the line and oblique lines be drawn;

1. *The perpendicular will be shorter than any oblique line.*
2. *Any two oblique lines which terminate at equal distances from the foot of the perpendicular are equal.*
3. *The oblique line which terminates at the greater distance from the foot of the perpendicular is the greater.*

Let C be a given point, and AD a given line, CE a perpendicular, and CA, CB, and CD, oblique lines; then,



First. In the triangle AEC, the angle AEC is a right angle, and, consequently, greater than A; therefore, the side CE is shorter than CA (Th. XII.).

Second. Let $AE = EB$; then, since CE is common and the angle $AEC = CEB$, the triangles AEC and CEB are equal (Th. VI.), and AC equals BC.

Third. Let ED be greater than EB; then, since CBE is an acute angle, CBD must be obtuse, and in the triangle CBD, CD is greater than BC, being opposite the greater angle (Th. XII.). Therefore, etc.

Cor. 1. It is evident that we cannot have two perpendiculars drawn from the same point to the same straight line.

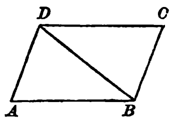
Cor. 2. If a straight line has two points equally distant from the extremities of a line, it will be perpendicular to that line. Let the pupil demonstrate it.

QUADRILATERALS.

THEOREM XV.

In any parallelogram, the opposite sides and angles are equal, each to each.

Let $ABCD$ be a parallelogram; then will AB be equal to DC, and AD to BC.



For, draw the diagonal DB. Then, since AB and DC are parallel, the alternate angles ABD and BDC are equal (Th. III.); and since AD and

BC are parallel, the alternate angles *ADB* and *DBC* are equal. Hence, the two triangles *ABD* and *DBC* have two angles and the included side, *DB*, of one, equal to two angles and the included side, *DB*, of the other, each to each; therefore, the triangles are equal (Th. VII.); and the side *AB* opposite the angle *ADB* is equal to the side *DC* opposite the equal angle *DBC:* hence, also, the side *AD* equals *BC;* therefore, the opposite sides of a parallelogram are equal.

Again, since the triangles are equal, the angle *A* is equal to the angle *C;* and the angle *ADC*, which is made up of the two angles *ADB* and *BDC*, is equal to the angle *ABC*, which is made up of the equal angles *DBC* and *ABD.* Therefore, etc.

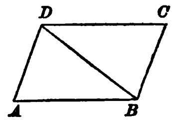
Cor. 1. The diagonal divides the parallelogram into two equal triangles.

Cor. 2. Two parallels included between two other parallels are equal.

Cor. 3. Two parallelograms are equal when they have two sides and the included angle of one, equal to two sides and included angle of the other.

THEOREM XVI.

If the opposite sides of a quadrilateral are equal, each to each, the equal sides are parallel, and the figure is a parallelogram.



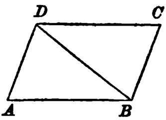
Let *ABCD* be a quadrilateral, in which *AB* equals *DC*, and *AD* equals *BC;* then will it be a parallelogram.

For, draw the diagonal *DB*. Then the triangles *ABD* and *DBC* have all the sides of the one equal to all the sides of the other, each to each; therefore the two triangles are equal (Th. IX.); and the

angle ABD opposite the side AD is equal to the angle BDC opposite the equal side BC; therefore, the side AB is parallel to the side DC (Th. IV.). For a like reason, AD is parallel to BC; therefore, the figure $ABCD$ is a parallelogram. Therefore, etc.

THEOREM XVII.

If two sides of a quadrilateral are equal and parallel, the figure is a parallelogram.

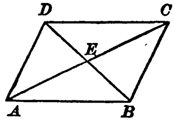


Let $ABCD$ be a quadrilateral, having the sides AB and DC equal and parallel; then will $ABCD$ be a parallelogram.

For, draw the diagonal DB. Then, since AB is parallel to DC, the alternate angles ABD and BDC are equal (Th. III.). Now, the triangles ABD and DBC have the side AB equal to DC, by hypothesis, the side DB common, and the included angles ABD and BDC equal; hence, the triangles are equal (Th. VI.), and the alternate angles ADB and DBC are equal; hence, the sides AD and BC are parallel (Th. IV.), and the figure is a parallelogram. Therefore, etc.

THEOREM XVIII.

The diagonals of a parallelogram bisect each other; that is, divide each other into equal parts.



Let $ABCD$ be a parallelogram, and AC and DB its diagonals; then will AE be equal to EC, and DE to EB.

For, since AB and DC are parallel, the angle CDE equals ABE (Th. III.); and also DCE equals EAB; and since AB equals DC, the triangles AEB and DEC have two angles and the included side of the one

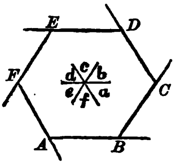
equal to two angles and the included side of the other; hence, the triangles are equal (Th. VII.), AE equals CE, and DE equals BE; therefore, the diagonals are bisected at E.

ANGLES OF POLYGONS.

THEOREM XIX.

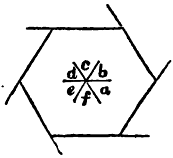
If the sides of a polygon be produced in the same direction, the sum of the exterior angles will be equal to four right angles.

Let $ABCDEF$ be a polygon, with the sides produced in the same direction; then will the sum of the exterior angles be equal to four right angles.



For, from any point within the polygon, draw lines respectively parallel to the sides of the polygon; the angles contained by the lines about this point will be equal to the exterior angles of the polygon (Th. V.). But the sum of the angles formed about a point equals four right angles (Th. II. C. 2); hence, the sum of the exterior angles of a polygon equals four right angles. Therefore, etc.

Cor. 1. The sum of the interior angles of a polygon is equal to twice as many right angles as the polygon has sides, less four right angles.



The sum of each exterior and interior angle equals two right angles, and there are as many of each as the polygon has sides; hence, the sum of all the exterior and interior angles equals two right angles taken as many times as there are sides of the polygon. But the sum of the exterior angles equals

four right angles; hence, the sum of the interior angles equals two right angles taken as many times as the polygon has sides, minus four right angles.

Cor. 2. The sum of the interior angles of a quadrilateral equals 2 right angles multiplied by 4, minus 4 right angles, which is $8-4$, or 4 right angles. In a rectangle each angle is a right angle.

Cor. 3. The sum of the angles of a pentagon equals $2\times 5-4=6$ right angles. Each angle of an equiangular pentagon is $\frac{1}{5}$ of 6 or $\frac{6}{5}$ of a right angle, or 108°.

Cor. 4. The sum of the angles of a hexagon equals $2\times 6-4=8$ right angles. Each angle of an equiangular hexagon is $\frac{4}{3}$ of a right angle, or 120°.

Cor. 5. In polygons of the same number of sides, the sum of the angles is equal. In equiangular polygons, each angle equals the sum divided by the number of sides.

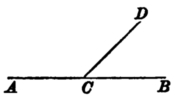
Scholium. This theorem must be restricted to convex polygons.

PRACTICAL EXAMPLES.

A common deficiency of pupils in the study of Geometry, is their inability to make a practical application of their knowledge. To remedy this, practical examples should be given, either in connection with the theorems or at the close of each book. The following problems may be used in either of these ways which the teacher may prefer.

1. If one line meet another line at an angle of 60°, what is the value of the adjacent angle?

SOLUTION.—If the line DC meets AB, making the angle DCB equal to 60°, the angle ACD will equal $180°-60°$, or 120°, since $ACD+DCB=180°$.



2. If two lines meet a third at the same point, making angles equal to 30° and 80° respectively, required the angle between the two lines.

3. How many degrees in each angle of a rectangle?

4. How many degrees in each angle of an equilateral triangle?

5. If two angles of a triangle are 43° and 75° respectively, what is the other angle?

6. If two angles of a triangle are each 45°, what is the other angle, and what is the kind of triangle?

7. If one angle of a triangle is 60°, what is each of the other two, if equal, and what is the kind of triangle?

8. If one of the two equal angles of a triangle is 30°, what is each of the other angles?

9. Required the number of degrees in each angle of an equiangular pentagon.

10. Required the number of degrees in each angle of an equiangular hexagon.

11. In a triangle whose angles are A, B, C, what is each angle if A is twice and B three times C?

12. In the preceding problem, what is the kind of triangle?

13. Required each angle of an isosceles triangle, if the unequal angle equals twice the sum of the other two.

14. Required the value of each exterior angle of an equiangular octagon.

EXERCISES FOR ORIGINAL THOUGHT.

We now give some theorems to exercise the pupil in original thought. The importance of such exercises cannot be overestimated. Much of the discipline of Geometry is lost by the pupil memorizing the demonstrations given in the book. One can become a good geometer only by trying his powers with new theorems and problems, and endeavoring to find out demonstrations and solutions for himself.

These theorems may be given upon review, one of them in connection with the regular lesson; or, if the teacher prefer, the lesson may consist wholly of them. With classes whose time for the study is limited, they may be omitted.

1. If the equal sides of an isosceles triangle be produced, the two obtuse angles below the base will be equal.

2. If the three sides of an equilateral triangle be produced, all the external acute angles will be equal, and all the obtuse angles will be equal.

3. Either side of a triangle is greater than the difference between the other two.

4. If a line be drawn bisecting an angle, any point of the bisecting line is equally distant from the sides of the angle.

5. Prove that the diagonals of a rectangle are equal.

6. If the diagonals of a quadrilateral bisect each other at right angles, the figure is a rhombus or square.

7. If a line joining two parallels be bisected, any other line through the point of bisection and joining the two parallels, is also bisected at that point.

8. If from any point within a triangle, two straight lines be drawn to the extremities of any side, their sum will be less than the sum of the other two sides of the triangle.

9. If a line is perpendicular to another line at its middle point,—1. Any point in the perpendicular will be equally distant from the extremities. 2. Any point out of the perpendicular will be unequally distant from the extremities.

BOOK II.

RATIO AND PROPORTION.

1. All reasoning is by comparison. In comparing two quantities, we see that they bear a certain relation to each other.

2. Ratio is the measure of the relation of two similar quantities. It is found by dividing the first by the second; thus, the ratio of 8 to 4 is $\frac{8}{4}$, or 2, the ratio of A to B is $\frac{A}{B}$

3. The two quantities compared are called the *Terms* of the ratio. The first is called the *Antecedent*, the second the *Consequent*, and the two constitute a *Couplet.*

4. A ratio is indicated by placing a colon between the quantities, or by writing the consequent under the antecedent, as in division; thus, the ratio of A to B is written, $A : B$, or $\frac{A}{B}$.

5. A Proportion is an expression of equality between equal ratios; thus, the ratio of 8 to 4 equals the ratio of 6 to 3, and a formal comparison of these, as $8 : 4 = 6 : 3$, is a proportion.

6. The equality of ratios is usually indicated by a double colon; thus, $8 : 4 :: 6 : 3$. This is read, the ratio of 8 to 4 equals the ratio of 6 to 3, or, 8 is to 4 as 6 is to 3.

7. There are four terms in a proportion; the first and fourth are called the *extremes;* the second and third, the

means. The first and second together are the *first couplet;* the third and fourth, the *second couplet.*

8. Quantities are in proportion by *Alternation,* when antecedent is compared with antecedent, and consequent with consequent; thus, if $A : B :: C : D$, by alternation we have $A : C :: B : D$.

9. Quantities are in proportion by *Inversion,* when the antecedents are made consequents and the consequents antecedents; thus, if $A : B :: C : D$, by inversion we have $B : A :: D : C$.

10. Quantities are in proportion by *Composition,* when the sum of antecedent and consequent is compared with either antecedent or consequent; thus, if $A : B :: C : D$, by composition we have, $A : A + B :: C : C + D$.

11. Quantities are in proportion by *Division,* when the difference of antecedent and consequent is compared with either antecedent or consequent; thus, if $A : B :: C : D$, we have, $A : A - B :: C : C - D$.

A Continued Proportion is one in which each consequent is the same as the next antecedent; as, $A : B :: B : C :: C : D$.

Analysis.—The object of the theorems of this book is to derive the principles of proportion. These principles are employed in the books which follow. The method consists in regarding a proportion as an *equation,* which it really is,—an *equality of ratios.* Thus, the pupil should be taught to regard the proportion $A : B :: C : D$ as equivalent to $A \div B = C \div D$, and as soon as this idea is clearly fixed in the mind the subject becomes simple and easy. The first proportion is the basis of demonstration for the others, and may be used as a test of the truth of all others.

THEOREM I.

If four quantities are in proportion, the product of the means will equal the product of the extremes.

Take the proportion

$A : B :: C : D$; then we wish to prove

that $\quad A \times D = B \times C.$

For, from the proportion we have

$\frac{A}{B} = \frac{C}{D}$; multiplying by $B \times D$,

we have, $\quad A \times D = B \times C.$

Therefore, *if four quantities are*, etc.

THEOREM II.

If the product of two quantities equals the product of two other quantities, two of them may be made the means, and two the extremes of a proportion.

Suppose we have

$A \times D = B \times C$; dividing by $B \times D$,

we have, $\quad \frac{A}{B} = \frac{C}{D}$; placing this in another form,

we have, $\quad A : B :: C : D.$

Therefore, etc.

THEOREM III.

A mean proportional between two quantities equals the square root of their product.

Let B be a mean proportional between A and C; then we have,

$$A : B :: B : C;$$

whence (Th. I.), $\quad B^2 = A \times C,$

or, $\quad B = \sqrt{A \times C}.$

Therefore, etc.

THEOREM IV.

If four quantities are in proportion, they will be in proportion by alternation.

Suppose $A : B :: C : D$; from this (Th. I.)
we have, $A \times D = B \times C$; dividing by $D \times C$,
we have, $\frac{A}{C} = \frac{B}{D}$; whence,

$A : C :: B : D$.

Therefore, etc.

REMARK.—The proposition is evidently true, since we have the same products when we take the product of the means and extremes as before the change. This principle may be applied to several other propositions.

THEOREM V.

If four quantities are in proportion, they will be in proportion by inversion.

Suppose $A : B :: C : D$; from this
we have, $\frac{A}{B} = \frac{C}{D}$; taking the reciprocal,
we have, $\frac{B}{A} = \frac{D}{C}$; whence,

$B : A :: D : C$.

Therefore, etc.

THEOREM VI.

If four quantities are in proportion, they will be in proportion by composition.

Suppose $A : B :: C : D$; then
we have, $\frac{A}{B} = \frac{C}{D}$. Adding *one* to each
we have, $\frac{A}{B} + 1 = \frac{C}{D} + 1$; reducing to a common denomi-

nator, we have, $\frac{A+B}{B}=\frac{C+D}{D}$; whence,

$A+B:B::C+D:D.$

Therefore, etc.

THEOREM VII.

If four quantities are in proportion, they will be in proportion by division.

Suppose $A:B::C:D$; then

we have, $\frac{A}{B}=\frac{C}{D}$; subtracting 1,

we have, $\frac{A}{B}-1=\frac{C}{D}-1$; reducing,

we have, $\frac{A-B}{B}=\frac{C-D}{D}$; whence,

$A-B:B::C-D:D.$

Therefore, etc.

THEOREM VIII.

If two proportions have a couplet in each the same, the other couplets will form a proportion.

Suppose $A:B::C:D$; and

$A:B::E:F$; then,

$\frac{A}{B}=\frac{C}{D}$ and $\frac{A}{B}=\frac{E}{F}$; hence (A. 1),

$\frac{C}{D}=\frac{E}{F}$; whence

we have, $C:D::E:F.$

Therefore, etc.

THEOREM IX.

Equimultiples of two quantities are proportional to the quantities themselves.

Let A and B be any two quantities; then

$\frac{A}{B}=\frac{A}{B}$; multiply both terms of the first by m,

we have, $\frac{mA}{mB}=\frac{A}{B}$; whence,

$$mA : mB :: A : B.$$

Therefore, etc.

THEOREM X.

If four quantities are in proportion, any equimultiples of the first couplet will be proportional to any equimultiples of the second couplet.

Suppose $A : B :: C : D$; then

$\frac{A}{B}=\frac{C}{D}$; hence, also,

we have, $\frac{mA}{mB}=\frac{nC}{nD}$; whence

we have, $mA : mB :: nC : nD.$

Therefore, etc.

THEOREM XI.

The products of the corresponding terms of two proportions are proportional.

Suppose $A : B :: C : D$, and

$M : N :: P : Q$; then

we have, $A \times D = B \times C$,

$M \times Q = N \times P$; taking their product,

we have,

$A \times M \times D \times Q = B \times N \times C \times P$; whence (Th. II.)

we have, $A \times M : B \times N :: C \times P : D \times Q.$

THEOREM XII.

If any number of quantities are in proportion, any antecedent will be to its consequent as the sum of the antecedents is to the sum of the consequents.

Let $A : B :: C : D :: E : F$, etc.

Then, since $A : B :: C : D$, and

$A : B :: E : F$; we have

$A \times D = B \times C$, and

$A \times F = B \times E$; adding

to these, $A \times B = A \times B$, we have,

$$A \times B + A \times D + A \times F = A \times B + B \times C + B \times E,$$

or, $A \times (B + D + F) = B (A + C + E)$;

whence, $A : B :: A + C + E : B + D + F$.

PRACTICAL EXERCISES.

1. If the first three terms of a proportion are 12, 14, and 18, what is the fourth term? *Ans.* 21.

2. Given the proportion $3 : 12 :: 5 : 20$; what proportion have we by composition?

3. Find a mean proportional to 12 and 27; to m and n. *Ans.* 18; $\sqrt{m \times n}$.

4. If the ratio of A to B is $\frac{1}{3}$, what is the ratio of $3A$ to $2B$? *Ans.* $\frac{1}{2}$.

5. If the ratio of $3A$ to $2B$ is $\frac{3}{4}$, what is the ratio of A to B? *Ans.* $\frac{1}{2}$.

6. What proportion is deducible from the equation $M \times N = A^2 - B^2$. *Ans.* $M : A + B :: A - B : N$.

7. What proportion is deducible from the equation $(C + D) \times A = (A + B) \times C$? *Ans.* $A : B :: C : D$.

THEOREMS FOR ORIGINAL THOUGHT.

1. If $a : b :: c : d$, prove that $am : bn :: cm : dn$.

2. If $a : b :: c : d$, prove that $\frac{a}{m} : \frac{b}{n} :: \frac{c}{m} : \frac{d}{n}$.

3. If $a : b :: c : d$, prove that $a : a + b :: c : c + d$.

4. If $a : b :: c : d$, prove that $a + b : a - b :: c + d : c - d$.

5. If $a : b :: c : d$ and $m : c :: n : d$, prove that $a : b :: m : n$.

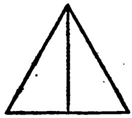
BOOK III.

AREAS AND RELATIONS OF POLYGONS.

1. THIS book treats of the area of polygons and their relation to each other.

2. The AREA of a polygon is its quantity of surface: it is expressed by the number of times which the polygon contains some other area assumed as a *unit of measure.*

3. The ALTITUDE OF A TRIANGLE is the perpendicular distance from the vertex of either angle to the opposite side, or the opposite side produced.

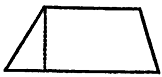


The vertex of the angle from which the altitude is drawn is called the *vertex of the Triangle;* the opposite side is called the *base* of the triangle.

4. The ALTITUDE OF A PARALLELOGRAM is the perpendicular distance between two opposite sides.

These opposite sides are called *bases,* one is the *upper base,* the other the *lower base.*

5. The ALTITUDE OF A TRAPEZOID is the perpendicular distance between its parallel sides.



These sides are called bases; one is called the *upper base,* the other the *lower base.*

6. SIMILAR POLYGONS are those which are mutually equiangular, and in which the sides containing the equal angles are proportional.

Homologous sides or *angles* are those which are like placed.

7. Equal Polygons are those which are equal in *area*. Polygons which, being applied to each other, coincide throughout their whole extent, are said to be *equal in all their parts.*

8. A Regular Polygon is a polygon which is both equilateral and equiangular.

Analysis.—The first object of this book is to find the area of polygons. It begins with the area of a rectangle, assuming as a unit of measure a square whose side is a measure of the sides of the given rectangle. From the area of the rectangle we pass to the area of any parallelogram, thence to the area of a triangle, and from this to the area of any plane figure.

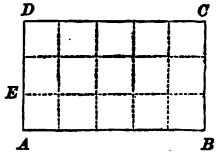
The book also treats of the relations of the squares on the sides of triangles, and the relation of the angles, sides, and area of similar polygons, to each other. It is one of the most interesting and practical books of Geometry.

AREA OF POLYGONS.

THEOREM I.

The area of a rectangle is equal to the product of its base and altitude.

Let $ABCD$ be a rectangle; then will its area be equal to the product of its base and altitude.



For, let the line AE be a unit of measure of the base and altitude, and suppose it contained any number as 5 times in the base and 3 times in the altitude; then, divide AB into 5 equal parts and AD into 3 equal parts, and through the points of division draw lines parallel, respectively, to the sides AB and AD; then will the rectangle be divided into equal squares. For, their sides are equal (B. I. Th. XV. C. 2); their angles are right (B. I. Th. III.); hence, the figures are equal squares (B. I. Th. XV. C. 2).

Now, the whole number of these squares is equal to the number in one row multiplied by the number of rows, which is the same as the number of linear units in the base multiplied by the number of linear units in the altitude; and the same is evidently true for any other numbers than 3 and 5. Hence, the area of $ABCD$ equals $AB \times AD$.

Since this is true when the linear unit of measure is any length, it is true when it becomes exceedingly small, and is, therefore, true when it becomes infinitely small, as it must when the two sides are incommensurable. There-

fore, the area of a rectangle is equal to the product of its base and altitude.

Cor. 1. *Rectangles are to each other as the product of their bases and altitudes.* For, let AB and AD represent the base and altitude of one rectangle, and EF and EH the base and altitude of another; then we will have the identical proportion, $ABCD : EFGH :: AB \times AD : EF \times EH$.

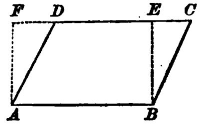
Cor. 2. *Rectangles having equal bases are to each other as their altitudes.* For, suppose the bases AB and EF equal; then, cancelling the equal factor in the second couplet, we have, $ABCD : EFGH :: AD : EH$.

Cor. 3. *Rectangles having equal altitudes are to each other as their bases.* For, suppose the altitudes AD and EH are equal; then, by cancelling the equal factor in the second couplet of Cor. 1, we have, $ABCD : EFGH :: AB : EF$.

THEOREM II.

The area of a parallelogram is equal to the product of its base and altitude.

Let $ABCD$ be a parallelogram, AB its base, and EB its altitude; then will its area be equal to $AB \times EB$.



For, at the points A and B draw the two perpendiculars AF and BE, and complete the rectangle $ABEF$. Then, the angle ADF equals the angle BCE, and FAD equals CBE (B. I. Th. V.); hence, the two triangles are equal (B. I. Th. VII.); therefore, $ABED + BCE$ is equal to $ABED + ADF$, or the parallelogram $ABCD$ is equal to the rectangle $ABEF$. But the area of the rectangle is equal to $AB \times BE$; hence, the area of the parallelogram is equal to $AB \times BE$. Therefore, etc.

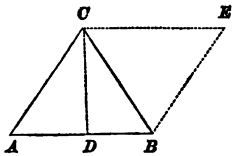
Cor. 1. Parallelograms are to each other as the product of their bases and altitudes.

Cor. 2. Parallelograms having equal altitudes are to each other as their bases; and parallelograms having equal bases are to each other as their altitudes.

THEOREM III.

The area of a triangle is equal to half the product of its base and altitude.

Let ABC be a triangle, AB its base, and CD its altitude; then will its area be equal to half the product of its base and altitude.



For, draw BE parallel to AC, and CE parallel to AB, completing the parallelogram $ABEC$; then will the triangle ABC be one-half the parallelogram $ABEC$ (B. I. Th. XV. C. 1). But the area of the parallelogram is equal to $AB \times CD$; hence, the area of the triangle is equal to $\frac{1}{2} AB \times CD$. Therefore, etc.

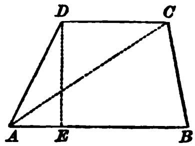
Cor. 1. Triangles are to each other as the product of their bases and altitudes.

Cor. 2. Triangles having equal altitudes are as their bases; having equal bases, they are as their altitudes.

THEOREM IV.

The area of a trapezoid is equal to one-half the sum of the parallel sides multiplied by the altitude.

Let $ABCD$ be a trapezoid, AB and DC its parallel sides, and DE its altitude; then will its area equal $\frac{1}{2}(AB + DC) \times DE$.

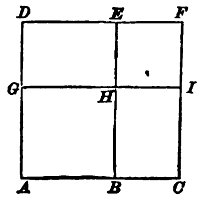


For, draw the diagonal AC, dividing the trapezoid into the two triangles ABC and ADC, the altitude of each being DE. The area of ABC is $\frac{1}{2} AB \times DE$, the area of ADC is $\frac{1}{2} DC \times DE$; hence, the area of $ABCD$, the sum of these triangles, is $\frac{1}{2} AB \times DE$ plus $\frac{1}{2} DC \times DE$, which is $\frac{1}{2}(AB + DC) \times DE$. Therefore, etc.

SQUARES ON LINES.

THEOREM V.

The square described on the sum of any two lines is equal to the sum of the squares described on the lines, plus twice the rectangle of the lines.



Let AB and BC be two lines, and AC their sum; then will

$$\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2 + 2\,AB \times BC.$$

For, on AC construct the square $ACFD$ and on AB construct the square $ABHG$; prolong BH to E and GH to I. Now, it is readily seen that $HIFE$ is the square of BC, also that $BCIH$ equals the rectangle on AB and BC, and $GHED$ equals the rectangle on AB and BC; therefore, the square $ACFD$ consists of the square on the two lines plus twice the rectangle contained by the two lines.

Cor. 1. *The square of the difference of two lines equals the sum of the squares of the lines, minus twice the rectangle of the lines.* AB is the difference of AC and BC; the figure $BCFE$ is the rectangle of AC and BC, and if we increase the rectangle GE by a square on BC, we will have another

rectangle of AC and BC; hence, the square on AC plus a square on BC equals twice the rectangle of AC and BC, plus the square of AB; or, $AB^2 = \overline{AC}^2 + \overline{BC}^2 - 2\,AC \times BC$.

Cor. 2. By a simple construction, somewhat similar to the above, it may also be shown that *the rectangle contained by the sum and difference of two lines equals the difference of their squares.*

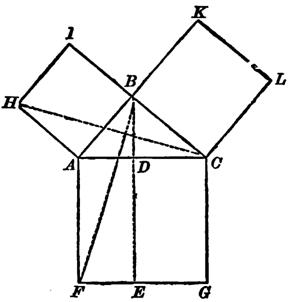
Scholium. These three principles are thus stated in algebraic language.

$$1.\ (a+b)^2 = a^2 + 2\,ab + b^2.$$
$$2.\ (a-b)^2 = a^2 - 2\,ab + b^2.$$
$$3.\ (a^2 - b^2) = (a+b)(a-b).$$

THEOREM VI.

The square described on the hypothenuse of a right-angled triangle is equal to the sum of the squares described on the other two sides.

Let ABC be a triangle, right-angled at B; then will $\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2$.



For, construct squares on each of the sides, draw BD parallel to AF and produce it to E, and draw the diagonals BF and HC. The two triangles HAC and BAF are equal; for, AC equals AF, being sides of the same square, HA equals AB, for the same reason, and the angle HAC equals the angle BAF, both being equal to a right angle plus BAC; hence, the triangle HAC equals BAF.

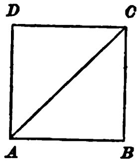
The triangle BAF is one-half of the rectangle $AFED$, since it has the same base and the same altitude (Th. III.);

also, since IBC is a straight line, the triangle HAC and square $ABIH$ have the same altitude; hence, the triangle is one-half of the square (Th. III.). But these two triangles BAF and HAC are equal; hence, the rectangle $AFED$ is equal to the square $ABIH$. In the same manner we may prove that the rectangle $EGCD$ is equal to the square $BCLK$; hence, the sum of the two rectangles, or the square on AC is equal to the sum of the two squares HB and BL. Therefore, etc.

Cor. 1. *The square of either side about the right angle is equal to the square of the hypothenuse diminished by the square of the other side.*

For, since $\overline{AB}^2 + \overline{BC}^2 = \overline{AC}^2$, we have, by transposing, $\overline{AB}^2 = \overline{AC}^2 - \overline{BC}^2$.

Cor. 2. *The square of the diagonal of a square is equal to twice the square of the side of the square.*



Let $ABCD$ be a square, then will $\overline{AC}^2 = 2\overline{AB}^2$. For, we have, by the theorem, $\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2$; but $\overline{AB}^2$ equals $\overline{BC}^2$; hence, by substitution, we have $\overline{AC}^2 = \overline{AB}^2 + \overline{AB}^2$, or, $\overline{AC}^2 = 2\overline{AB}^2$.

Cor. 3. *The side of a square is to its diagonal as* 1 *is to the square root of* 2.

For, since $2\overline{AB}^2 = \overline{AC}^2$, or, $2 \times \overline{AB}^2 = \overline{AC}^2 \times 1$, we have the proportion (B. II. Th. II.),

$$\overline{AB}^2 : \overline{AC}^2 :: 1 : 2; \text{ extracting the square root,}$$

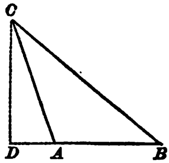
we have, $AB : AC :: 1 : \sqrt{2}$.

NOTE.—This is the celebrated Pythagorean proposition, so called because it was discovered by Pythagoras. It is also known as the 47th of Euclid, that being the number of the proposition in the first book of Euclid's Elements.

THEOREM VII.

In any obtuse-angled triangle, the square of the side opposite the obtuse angle is equal to the sum of the squares of the other two sides, plus twice the product of the base into the distance from the vertex of the obtuse angle to the foot of the perpendicular drawn from the vertex of the opposite angle to the base produced.

Let ABC be a triangle, of which A is an obtuse angle, AB its base, and CD the perpendicular drawn to the base produced; then will



$$\overline{BC}^2 = \overline{AC}^2 + \overline{AB}^2 + 2\,AB \times AD.$$

For, in the right-angled triangle DBC, we have, $\overline{BC}^2 = \overline{DC}^2 + \overline{DB}^2$;

but $\quad DB = AB + AD$;

hence, $\quad \overline{DB}^2 = \overline{AB}^2 + \overline{AD}^2 + 2\,AB \times AD$ (Th. V.).

Hence, $\quad \overline{BC}^2 = \overline{DC}^2 + \overline{AB}^2 + \overline{AD}^2 + 2\,AB \times AD$.

But, $\quad DC^2 + \overline{AD}^2 = \overline{AC}^2$.

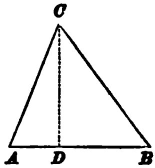
Hence, $\quad BC^2 = \overline{AB}^2 + \overline{AC}^2 + 2\,AB \times AD$.

Cor. 1. If the angle CAB becomes a right angle, AD becomes zero, and we have, $\overline{BC}^2 = \overline{AB}^2 + \overline{AC}^2$.

THEOREM VIII.

In any triangle, the square of a side opposite an acute angle is equal to the sum of the squares of the other two sides, minus twice the product of the base and the distance from the vertex of the acute angle to the foot of the perpendicular let fall upon the base or the base produced.

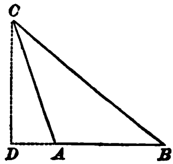
Let ABC be any triangle, B an acute angle, AB its base, and CD the perpendicular; then will



$$AC^2 = \overline{AB}^2 + \overline{BC}^2 - 2\,AB \times BD.$$

For, in the right-angled triangle ADC,

we have, $\overline{AC}^2 = \overline{DC}^2 + \overline{AD}^2$;
but $AD = AB - DB$;
hence, $\overline{AD}^2 = \overline{AB}^2 + \overline{DB}^2 - 2\,AB \times DB$ (Th. V. C. 1).
Hence, $\overline{AC}^2 = \overline{DC}^2 + \overline{AB}^2 + \overline{DB}^2 - 2\,AB \times DB$.
But, $\overline{DC}^2 + \overline{DB}^2 = \overline{BC}^2$, in BDC.
Hence, $\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2 - 2\,AB \times DB$.



The same may also be shown if the perpendicular meets the base produced, as in the second figure. Therefore, etc.

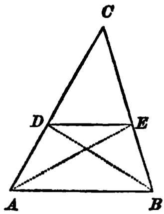
NOTE.—This 8th Proposition can be very prettily drawn from the 7th, by transposing the terms of the 7th, and reducing. Let the pupil try it.

THEOREM IX.

In any triangle, a straight line drawn parallel to the base divides the other sides proportionally.

Let ABC be a triangle, and DE a line parallel to the base; then will

$$CD : DA :: CE : EB.$$



For, draw AE and DB; then, since the two triangles ADE and DEC have their bases in the same line and their vertices at the same point E, they have the same altitude; hence, they are to each other as their bases (Th. III. C. 2), or,

$$AED : DEC :: AD : DC.$$

For a similar reason, the triangles BED and DEC are to each other as their bases; hence, we have,

$$BED : DEC :: BE : EC.$$

But the triangles AED and BED have the same base DE and the same altitude, since their vertices are in the line AB parallel to DE; hence, they are equal (Th. III.), and

the two proportions have a couplet in each equal; hence, the remaining terms are proportional (B. II. Th. VIII.), and we have,

$$AD : DC :: BE : EC.$$

Therefore, etc.

Cor. 1. By composition, we have,

$$AD + DC : AD :: BE + EC : BE,$$

or, $AC : AD :: BC : BE$; and, in the same way,

$$AC : DC :: BC : EC.$$

Cor. 2. Conversely, *If a line divides two sides of a triangle proportionally, it will be parallel to the third side.*

For, if a line passing through D were any other than DE, it would not divide the lines proportionally, since the line BC is cut proportionally at the point E; hence, the line must be DE, a line parallel to the base.

Cor. 3. Since $DEC : AEC :: DC : AC$ and $AEC : ABC : EC : BC$, and also, $DC : AC :: EC : BC$; therefore,

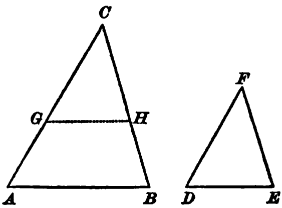
$$DEC : AEC :: AEC : ABC.$$

That is, *the triangle AEC is a mean proportional between DEC and ABC.*

SIMILAR TRIANGLES.

THEOREM X.

Equiangular triangles have their homologous sides proportional, and are similar.



Let ABC and DEF be two triangles having the angle $A = D$, the angle $B = E$, and $C = F$; then will they be similar.

For, on AC take CG equal to FD, and on BC take CH equal to

FE, and draw *GH;* then the triangle *CGH* will be equal to *FDE* (B. I. Th. VI.) and the angle *CGH* will equal *FDE;* hence, the angle *CGH* equals *CAB*, and *GH* is parallel to *AB* (B. I. Th. IV.). Hence, we have (Th. IX. C. 1).

$$AC : BC :: GC : HC, \text{ or,}$$

$$AC : BC :: DF : EF;$$

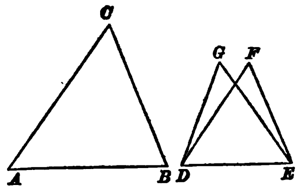
and the same may be shown for the sides containing the other equal angles; hence, the triangles are similar (D. 6). Therefore, etc.

THEOREM XI.

Triangles which have their corresponding sides proportional are equiangular and similar.

Let *ABC* and *DEF* be two triangles having their corresponding sides proportional; then will they be equiangular.

For, if they are not equiangular, suppose some other triangle, as *DEG*, to be constructed upon the side *DE*, equiangular with *ABC*. Then, by the preceding theorem, we have,



$$AB : DE :: AC : DG;$$

but, by hypothesis,

$$AB : DE :: AC : DF; \text{ hence,}$$

we have, $$DG = DF.$$

In the same way, it may be shown that

$$EG = EF.$$

Hence, the triangle *DEG* must be identical with the triangle *DEF* (B. I. Th. IX.), and, therefore, cannot be different from *DEF;* hence, *ABC* and *DEF* are equiangular, and, consequently, similar. Therefore, etc.

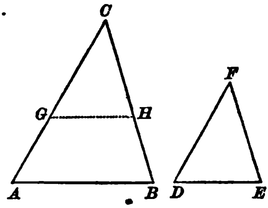
THEOREM XII.

Triangles which have an angle in each equal, and the sides including them proportional, are similar.

Let ABC and DEF be two triangles having the angle C equal to the angle F, and

$$AC : BC :: DF : EF;$$

then will the triangles be similar.



For, apply the angle DFE to ACB, and the triangle DFE will take the position GCH, and, from the proportion above, we shall have

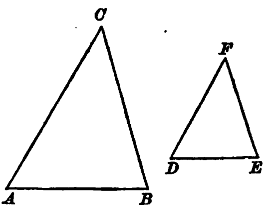
$$AC : BC :: GC : HC;$$

hence, GH is parallel to AB (Th. IX. C. 2), and the triangles GCH and ACB equiangular, and, therefore, similar. But, GCH is equal to DFE; therefore, ACB and DFE are equiangular, and, consequently, similar. Therefore, etc.

THEOREM XIII.

Triangles which have their sides parallel, each to each, or perpendicular, each to each, are similar.

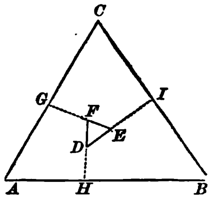
First. Let ABC and DEF be two triangles having the side AB parallel to DE, AC parallel to DF, and CB parallel to FE; then will they be similar.



For, since AC is parallel to DF and AB to DE, the angle A is equal to D (B. I. Th. V.); for a similar reason C is equal to F and B to E; hence, the triangles are mutually equiangular, and, consequently, similar.

Second. Let ABC and DEF be two triangles having their sides respectively perpendicular; then will they be similar.

For, produce the sides of DEF till they meet the sides of ABC. In the trapezium $GEIC$, the sum of the four angles equals four right angles (B. I. Th. XIX. C. 2), and since two of the angles are right angles, the sum of the angles C and GEI equals two right angles. But the sum of GEI and FED equals two right angles (B. I. Th. I.); hence, the angle FED equals the angle C. In the same way it may be shown that FDE equals B, and DFE equals A; hence, the two triangles are mutually equiangular, and, consequently, similar. Therefore, etc.



THEOREM XIV.

If, in a right-angled triangle, a line be drawn from the vertex of the right angle perpendicular to the hypothenuse;

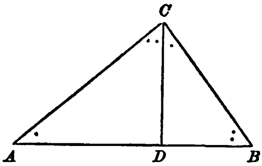
1. *The two triangles thus formed will be similar to the given triangle and to each other.*

2. *Each side about the right angle will be a mean proportional between the hypothenuse and adjacent segment.*

3. *The perpendicular will be a mean proportional between the two segments of the hypothenuse.*

Let ABC be a right-angled triangle, C the right angle, and CD the perpendicular; then,

First. The triangles ACD and ABC have each a right angle, and the angle A common; hence, the remaining angles are equal, and the triangles are similar (Th. X.).



In the same manner, we show BCD and ABC equiangular and similar; and then ADC and BDC, being both similar to ABC, are similar to each other.

Second. The two triangles being similar to the given one, we have,

$$AB : AC :: AC : AD,$$

and also, $$AB : BC :: BC : BD.$$

Therefore, etc.

Third. The two triangles being similar, we have,

$$AD : DC :: DC : DB.$$

Therefore, etc.

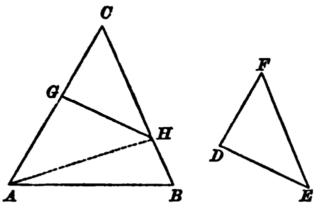
RELATION OF POLYGONS.

THEOREM XV.

Triangles which have an angle in each equal, are to each other as the product of the sides including those equal angles.

Let ABC and DEF be two triangles having the angle F equal to the angle C; then will

$$ABC : DEF :: AC \times BC : DF \times EF.$$



For, place the angle F on its equal C, and the triangle DEF will take the place GCH; then draw AH. Now, since the triangles AHC and GHC have their bases AC and GC in the same line AC, and vertices at H, they have the same altitude, and are to each other as their bases; hence,

$$AHC : GHC :: AC : GC.$$

Also, since AHC and ABC have their bases HC and BC in the same line, and vertices at the point A, they have the same altitude, and are as their bases; hence,

$$ABC : AHC :: BC : HC;$$

multiplying the corresponding terms of these two proportions together, and omitting the common factor AHC, we have,

$$ABC : GHC :: AC \times BC : GC \times HC,$$

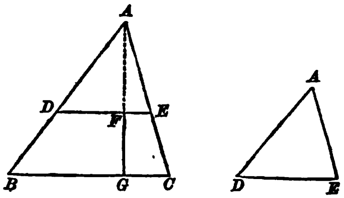
or, $$ABC : DEF :: AC \times BC : DF \times FE.$$

Therefore, etc.

THEOREM XVI.

Similar triangles are to each other as the squares of their homologous sides.

Let ABC and ADE be two similar triangles; then will they be to each other as the squares of any two homologous sides. Draw the altitudes AG and AF; then, since the triangles are as the product of their bases and altitudes (Th. III. C. 1), we have,



$$ABC : ADE :: BC \times AG : DE \times AF.$$

But, by similar triangles, we have,

$$BC : DE :: AB : AD,$$

and, $$AG : AF :: AB : AD;$$

hence, $$BC \times AG : DE \times AF :: \overline{AB}^2 : \overline{AD}^2.$$

Comparing this with the first proportion, we have,

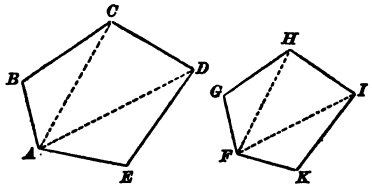
$$ABC : ADE :: \overline{AB}^2 : \overline{AD}^2.$$

THEOREM XVII.

Similar polygons may be divided into the same number of triangles, similar each to each, and similarly situated.

Let $ABCDE$ and $FGHIK$ be two similar polygons, having the angle A equal to the angle F, B to G, C to H, etc.; then

can they be divided into the same number of similar triangles similarly situated.



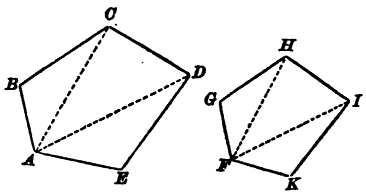
From the homologous angles A and F draw the diagonals AC, AD, and FH, FI. Since the polygons are similar, the triangles ABC and FGH have the angles B and G equal, and the sides about these angles proportional; they are, therefore, similar (Th. XII.).

Since the triangles ABC and FGH are similar, the angle ACB equals FHG, and the sides AC and FH are proportional to BC and GH, and hence to CD and HI. If we take the equal angles ACB and FHG from the equal angles BCD and GHI, we have ACD equal to FHI; hence, the triangles ACD and FHI have an angle in each equal, and the sides including these angles proportional; they are, therefore, similar (Th. XII.). In a similar manner, it may be shown that ADE and FIK are similar. Therefore, etc.

THEOREM XVIII.

The perimeters of similar polygons are to each other as any two homologous sides; and the polygons are to each other as the squares of those sides.

Let $ABCDE$ and $FGHIK$ be two similar polygons; then will their perimeters be to each other as any two homologous sides, and their areas be as the squares of those sides.



First. Since the polygons are similar, we have,

$$AB : FG :: BC : GH :: CD : HI, \text{ etc.};$$

hence (B. II. Th. XII.),

$$AB + BC + CD + \text{etc.} : FG + GH + HI + \text{etc.} :: AB : FG;$$

or, the perimeter of the first to the perimeter of the second as any side of the first to the homologous side of the second.

Second. Since the triangles are respectively similar, we have, $ABC : FGH :: \overline{AC}^2 : \overline{FH}^2;$

and also, $ACD : FHI :: \overline{AC}^2 : \overline{FH}^2;$

hence, we have, $ABC : FGH :: ACD : FHI.$

In a similar manner, we find,

$$ACD : FHI :: ADE : FIK.$$

Hence (B. II. Th. XII.), the sum of the antecedents, $ABC + ACD + ADE$, is to the sum of the consequents, $FGH + FHI + FIK$, as any antecedent ABC is to its consequent FGH; and, since ABC is to FGH as $\overline{AB}^2$ to $\overline{FG}^2$, we have,

$$ABCDE : FGHIK :: \overline{AB}^2 : \overline{FG}^2.$$

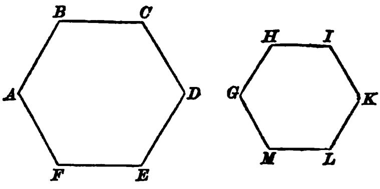
Therefore, etc.

Cor. The perimeters are to each other as any two *homologous lines*, and the polygons are as the squares of those lines.

THEOREM XIX.

Regular polygons of the same number of sides are similar figures.

Let $ABCDEF$ and $GHIKLM$ be two regular polygons of the same number of sides; then will they be similar.



For, the corresponding angles in each are equal (B. I. Th. XIX. C. 5), and the corresponding sides are proportional, since they

are equal; hence, the polygons are similar (D. 6). Therefore, etc.

Cor. Since regular polygons of the same number of sides are similar figures, their perimeters are proportional to any homologous lines, and their areas are as the squares of those lines.

PRACTICAL EXAMPLES.

1. Required the perimeter and area of a square whose sides are each 20 inches.

2. Required the perimeter and area of a rectangle whose sides are respectively 18 and 24 inches.

3. What is the area of a parallelogram whose base is 16 inches and altitude 12 inches?

4. A man has a board in the form of a triangle; what is its area if the base is 9 feet and the altitude 18 inches?

5. A farmer has a field in the form of a trapezoid; the two parallel sides are 40 and 60 rods, and the distance between them 32 rods; required its area.

*6. Required the hypothenuse of a right-angle triangle, the two sides being 3 and 4 inches respectively.

7. The sides of a triangle are 18 and 21, and the base 24; what are the sides of a similar triangle whose base is 8?

8. A man had a lot in the form of a right-angle triangle; the hypothenuse is 78 and one side 30; required the other side and the area.

9. A ladder 65 feet long is placed against a house, so that its foot is 25 feet from the house; how high does it reach?

10. A pole was broken 75 feet from the top, and fell so that the end struck 60 feet from the foot; required the length of the pole.

11. A has a triangular piece of ground, the base of the triangle being

---

* The numbers 3, 4, and 5 are the smallest integers which can express the relation of the three sides of a right-angle triangle. It is evident that we may have an infinite number of right-angle triangles with their sides in this ratio. Thus, 6, 8, 10; 9–12, 15, etc. Another integral relation of sides is 5, 12, 13.

20 rods; what is the base of a similarly-shaped lot containing 4 times as much land? *Ans.* 40 rods.

12. A man has a lot 40 rods long and 23 rods wide; what are the dimensions of a similar lot 9 times as large? *Ans.* 120; 69.

13. A ladder, whose length is 91 feet, star
how far must it be drawn out at the botto
ered 7 feet?

14. A ladder 130 feet long, with its foot
one side to a window 78 feet high, and on t
high; what is the width of the street?

15. There is a rectangular field whose
yards respectively; what is the side of a sq

16. If it cost $328 to put a fence around
rods wide, how much less will it cost to encl
area with the same kind of fence?

17. The gable ends of a house are each 48
dicular height of the ridge above the eaves is
boards will it take to board up both gables?

18. A man has a field in the form of a re
acres; what are its dimensions if the length
Ans. Length, 113.136 r

19. A cemetery containing 60 acres is laid
its length is equal to three times its width;
the cemetery. *Ans.* Length, 169.704 r

20. A general wishing to draw up his corp
found by the first trial he had 100 men over
side of the square by 2 men, and found he la
the square; how many men had he in the co

21. A man has a square yard containing
gravel walk around it which occupies $\frac{1}{64}$ of th
width of the walk?

22. In a triangle the two sides are 13 an
perpendicular from the vertex of the angle
posite side, 12; required the third side. *Ans.* 14.

EXERCISES FOR ORIGINAL THOUGHT.

1. Two squares are to each other as the squares of their diagonals.

2. Two similar parallelograms are to each other as the squares of their diagonals.

3. Prove that the diagonals of a rectangle are equal to each other.

4. Prove that the greater diagonal of a parallelogram is opposite the greater angle.

5. Show where a line from the vertex of a triangle must be drawn to divide the triangle into two equal parts.

6. Prove that the ratio of the side of a square to its diagonal is as 1 to the square root of 2.

7. The straight line joining the middle points of the oblique sides of a trapezoid will be parallel to the other sides, and equal to half their sum.

8. The four lines joining the middle points of the adjacent sides of a quadrilateral form a parallelogram.

9. The lines drawn from the vertices of the three angles of an equilateral triangle, perpendicular to the opposite sides of the triangle, will intersect each other in the same point.

10. The line which bisects the vertical angle of a triangle divides the base into two parts which are proportional to the adjacent sides.

11. If a line be drawn parallel to the base of a triangle, and lines be drawn from the vertex of the triangle to the base, these lines will divide the base and parallel proportionally.

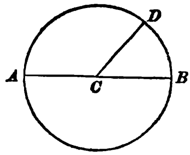
12. Triangles which have an angle in each equal, are to each other as the rectangles of the sides including those angles.

BOOK IV.

OF THE CIRCLE.

DEFINITIONS.

1. A **Circle** is a plane bounded by a curve line, every point of which is equally distant from a point within, called the *centre*.

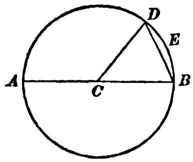


2. The **Circumference** is the bounding line of a circle. An **Arc** is any part of the circumference; as, *BD*.

3. The **Radius** is a straight line drawn from the centre to any point of the circumference; thus, *CD* is a radius.

4. The **Diameter** is a straight line passing through the centre and terminating at both extremities in the circumference; as, *AB*.

5. A **Chord** is a straight line joining the extremities of an arc; thus, *BD* is a chord.

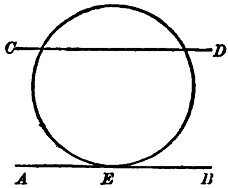


6. A **Segment** is a portion of the circle included between an arc and its chord; as, *DBE*.

7. A **Sector** is a portion of the circle included by an arc and the radii drawn to its extremities; as, *DCBE*.

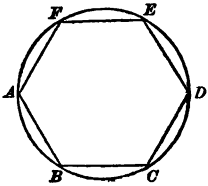
8. A **Tangent** is a straight line which touches the circumference in one point; thus, *AB* is a tangent. The point *E* is called the *point of tangency*.

9. A Secant is a straight line which cuts the circumference in two points; thus, CD is a secant.



10. An Inscribed Angle is an angle whose vertex is in the circumference and whose sides are chords; as, ABC in the next figure.

11. An Inscribed Polygon is a polygon whose sides are chords, the vertices of the angles being in the circumference; as, $ABCDEF$.



12. A Polygon is *circumscribed about a circle* when all of its sides are tangents to the circumference. The circle is at the same time *inscribed in a polygon.*

AXIOMS.

1. The radii, and also the diameters, of a circle, or of equal circles, are equal.

2. Every diameter is double the radius, or is equal to the sum of two radii.

3. A straight line can cut a circumference in only two points.

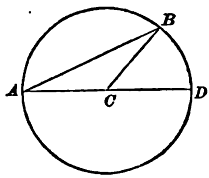
Analysis.—This book treats of the nature of the circle, the measurement of angles, the finding of the circumference, the measurement of the area of a circle, and the relation of the circumferences; and also of the areas of circles. The method of treatment in finding the circumference and area, and also their relations, is to regard the circle as a polygon of an infinite number of sides, and derive the principles from those of polygons. By a simplification of the subject, we embrace in one book what is usually given in two.

NATURE OF THE CIRCLE.

THEOREM I.

The diameter of a circle is greater than any chord.

Let AB be any chord; then will it be less than any diameter.

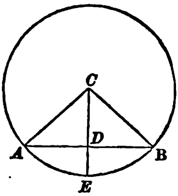


For, from the point A draw the diameter AD, and also draw the radius CB. Then, in the triangle ACB, the sum of the sides AC and CB is greater than AB (B. I. A. 10. C.). But $AC + CB$ equals AD (Ax. 2); hence, AD is greater than AB. Therefore, etc.

THEOREM II.

Any radius which is perpendicular to a chord bisects the chord and also the arc subtended by the chord.

Let AB be the chord, and CD the radius perpendicular to it; then will $AD = DB$ and $AE = EB$.



First. Draw the radii CA and CB; then the angle ACD equals DCB (B. I. Th. X. C. 1), and the triangles ACD and DCB are equal (B. I. Th. VI.); hence, the side AD equals DB.

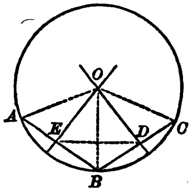
Second. Since the triangles ACD and DCB are equal, the angle ACE equals ECB; hence, if we apply the sector ACE to the sector ECB, the sides will coincide, and the arcs AE and EB will also coincide, since

each point of them is equally distant from the centre; hence, the arcs are equal. Therefore, etc.

THEOREM III.

Through three points not in the same line a circumference may be made to pass.

Let A, B, and C be any three points not in the same straight line; then may a circumference be described through them.



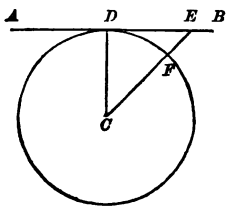
Draw AB and BC, and at E and D, the middle points of AB and BC, draw perpendiculars, and unite the points E and D. Now, since $OED + ODE$ is less than two right angles, the perpendiculars will meet in some point, as O. Draw OA, OB, and OC; then $OA = OB$ (B. I. Th. XIV.), and, for the same reason, $OB = OC$; hence, a circumference described from O as a centre will pass through the three points A, B, and C. Therefore, etc.

Cor. It may also be readily shown that but one circumference can be made to pass through three points.

THEOREM IV.

If a straight line is perpendicular to a radius at its extremity, it will be tangent to the circle at that point.

Let the straight line AB be perpendicular to the radius CD at D; then will it be tangent to the circle at the point D.



For, take any point of AB, as E, and draw the line CE. Now, CE is greater than CD (B.I.Th.XIV.); consequently, the point E will be without the circle, and hence the line AB

touches the circumference in only one point: it is therefore tangent to it at the point D (D. 8). Therefore, etc.

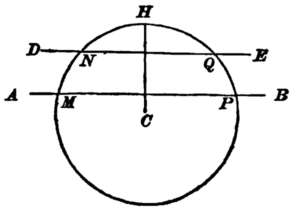
Cor. Conversely.—*A tangent to the circle is perpendicular to the radius drawn to the point of contact.*

For any line, as CE, is greater than CF, or its equal CD; hence, CD, being the shortest line from C to the tangent, is perpendicular to the tangent at D (B. I. Th. XIV.) Therefore, etc.

THEOREM V.

Two parallel lines intercept equal arcs on the circumference.

Let AB and DE be two lines cutting the circle; then will the arcs MN and PQ be equal.



Draw the radius CH perpendicular to the chord NQ; it will be perpendicular to MP (B. I. Th. III. C.), and H will be at the middle point of the arc NHQ, and also of the arc MHP (Th. II.); hence, NH equals HQ, and MH equals HP, and MN, which is the difference of the arcs MH and NH, is equal to PQ, which is the difference of the equal arcs PH and QH. Therefore, etc.

Cor. Since the theorem is true for the secant lines in any position, it is evidently true when either or both of the secants become tangents. Let the pupil illustrate these two cases.

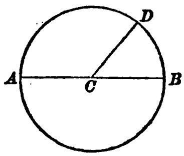
MEASUREMENT OF ANGLES.

THEOREM VI.

An angle having its vertex at the centre of a circle is measured by the arc included between its sides.

This proposition is assumed in the definition of Book I. as a fact; it will now be shown that such an assumption is authorized.

Suppose the radius *CB* to revolve around to the position *CD;* the difference of direction or divergence of the two lines *CD* and *CB* must depend upon the amount of the movement of *CD*, and this is evidently measured by the distance passed over by the extremity *D;* but the distance passed over by *D* is the arc *BD;* hence, the arc *BD* may be assumed as the measure of the angle *BCD*. Therefore, etc.



Scholium. The general method of demonstrating this theorem is to prove, first, that in the same circle, or in equal circles, angles at the centre are to each other as their intercepted arcs, and from this infer that the arc may be taken as the measure of the angle. A few authors assume the measure as an axiom.

To measure an angle, it would seem natural to assume a quantity of the same kind; and for this purpose the right angle would naturally be taken as the unit of measure. It has been found most convenient, however, to assume the circumference as the measure of an angle; and for this

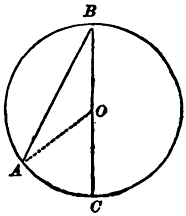
purpose it has been divided into degrees, minutes, and seconds, as before explained.

THEOREM VII.

An angle at the circumference of a circle is measured by half the arc included between its sides.

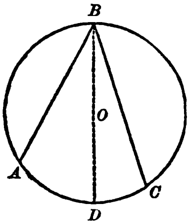
There may be three cases; first, when the centre of the circle is on one of the sides of the angle; second, when it is within the angle; third, when it is without the angle.

First. Let ABC be the angle, having its vertex at B, and O be the centre of the circle; then will ABC be measured by one-half of AC.



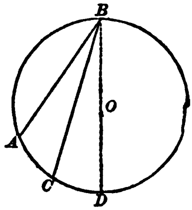
For, draw the radius AO; then the exterior angle AOC is equal to the sum of the opposite interior angles ABO and OAB (B. I. Th. XIII. C. 5). But the triangle AOB is isosceles; hence, the angles A and B are equal, and, consequently, the angle AOC is double the angle ABC. But AOC, being at the centre, is measured by the arc AC (Th. VI.); hence, the angle ABC is measured by one-half of the arc AC.

Second. Let ABC be the angle, and O the centre of the circle; then will ABC be measured by one-half of ADC.



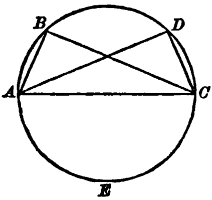
For, draw the diameter BD; then, from what we have just shown, the angle ABD is measured by one-half of AD, and the angle DBC by one-half of DC; hence, their sum, or the angle ABC, is measured by one-half of the sum of AD and DC, or one-half of ADC.

Third. Let ABC be the angle, and O the centre being without the angle; then will ABC be measured by one-half of AC.

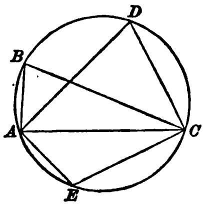


For, draw the diameter BD; then, since ABD is measured by one-half of AD, and CBD by one-half of CD, ABC, their difference, is measured by one-half of AD minus CD, or one-half of AC. Therefore, etc.

Cor. 1. All the angles ABC, ADC, inscribed in a semicircle are equal, being measured by one-half of the semicircumference AEC (Th. VI.)



Cor. 2. All the angles ABC, ADC, etc., inscribed in a segment greater than a semicircle are less than right angles, being measured by less than one-half of a semi-circumference. Any angle AEC inscribed in less than a semicircle is greater than a right angle, being measured by more than one-half of a semi-circumference.



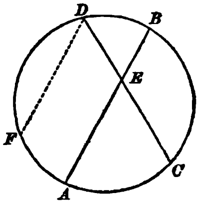
Cor. 3. All the angles inscribed in the same segment are equal, being measured by one-half of the same arc.

THEOREM VIII.

An angle formed by two chords which intersect is measured by half the sum of the included arcs.

Let AEC be an angle formed by the intersection of the chords AB and CD; then will it be measured by half of $AC + DB$.

For, draw DF parallel to AB; then the arc AF equals the arc DB (Th. V.), and the angle FDC equals the angle AEC (B. I. Th. III.). Now, the angle FDC is measured by one-half the arc FC (Th. VII.); hence, the angle AEC is measured by one-half of FC, or by one-half of $AC + AF$, or one-half of $AC + DB$. Therefore, etc.

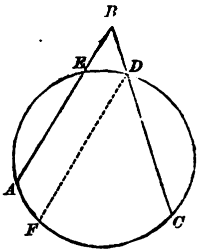


THEOREM IX.

The angle formed by two secants is measured by half the difference of the included arcs.

Let the angle ABC be formed by the two secants AB and CB; then will it be measured by one-half the difference of the arcs AC and ED.

For, draw DF parallel to AB; then the arc AF is equal to the arc ED, and the angle FDC equal to ABC. Now, the angle FDC is measured by one-half of the arc FC; hence, ABC is measured by one-half of FC; that is, by one-half of AC minus AF, or one-half of AC minus ED.



Cor. 1. *The angle formed by a secant and tangent is measured by one-half of the difference of the included arcs.* For, the theorem being true for two secants, it is also true when one of the secants becomes a tangent.

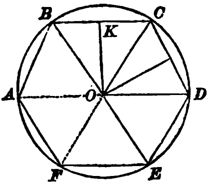
Cor. 2. *The angle formed by a tangent and a chord at the point of contact is measured by one-half of the included arc.* For, if one of the secants becomes a tangent and the other a chord at the point of tangency, there will be but one arc included between them, and the angle will, therefore, be measured by one-half of this arc.

THE CIRCUMFERENCE AND AREA.

THEOREM X.

The circumference of a circle may be circumscribed about a regular polygon, and it may also be inscribed within it.

Let $ABCD$ be a regular polygon; then can the circumference of a circle be circumscribed about it.



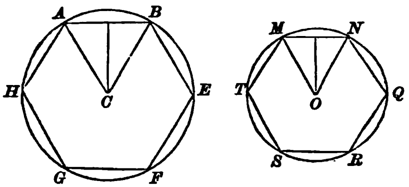
Through the three vertices A, B, and C, describe a circumference; its centre O will be in OK drawn perpendicular to BC at its middle point K. The triangles AOB and BOC being mutually equilateral are equal; hence, the angle ABO equals CBO, consequently, OB bisects ABC; and since OCB equals OBC, OC must bisect BCD, which is equal to ABC; hence, the triangles OCB and OCD have two sides and an included angle respectively equal, and are equal (B. I. Th. VII.), and OD equals OB; hence, the circumference passing through B also passes through D; and in the same way it may be shown to pass through all the vertices.

Second. Since the triangles AOB, BOC, etc. are all equal, their altitudes are equal; hence, a circumference described from O as a centre with the radius OK will touch all the chords at their middle points, and, consequently, be inscribed within the polygon. Therefore, etc.

THEOREM XI.

The circumferences of circles are as their radii, and their areas are as the squares of their radii.

Let C and O be the centres of two circles whose radii are CA and OM; then will their circumferences be to each other as their radii, and their areas as the squares of their radii.



Inscribe in the circles regular polygons of the same number of sides. These polygons being similar figures, their perimeters are to each other as any two homologous lines CA and OM, and their areas are as the squares of those lines (B. III. Th. XVIII. C.); and this is true whatever the number of sides; hence, it is true if the number of sides is infinite, and the polygon becomes the circle. Hence, we have,

$$\textit{circ. } CA : \textit{circ. } OM :: CA : OM; \text{ and, also,}$$
$$\textit{area } CA : \textit{area } OM :: \overline{CA}^2 : \overline{OM}^2.$$

Cor. 1. Since the radii of circles are to each other as the diameters, we have the circumferences to each other as the diameters, and the areas as the squares of the diameters.

Cor. 2. From this we see that the circumference of a circle is to its diameter as the circumference of another circle to its diameter; hence, the ratio of the circumference to the diameter is a constant quantity. This constant ratio mathematicians represent by π, the Greek letter

p, called *pi*. Letting C represent the *circumference* and D the *diameter* we have $\pi = \frac{C}{D}$.

NOTE.—This symbol π is of great importance in mathematics: the pupil should be very careful to thoroughly understand its signification and use.

THEOREM XII.

The circumference of a circle equals the diameter multiplied by π.

Since the ratio of the circumference to the diameter is represented by π, we have,

$$\frac{C}{D} = \pi; \text{ and, multiplying by } D,$$

we have, $$C = \pi . D.$$

Therefore, etc.

Cor. Since the diameter is twice the radius, if we substitute 2 R for D, we will have,

$$C = \pi \times 2\,R, \text{ or } C = 2\,\pi R.$$

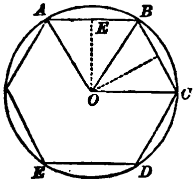
Hence, the circumference equals the radius multiplied by 2 π.

REMARK.—The value of π cannot be exactly expressed in numbers. The number generally used is 3.1416, which is sufficiently accurate for practical purposes.

THEOREM XIII.

The area of a circle is equal to the circumference multiplied by one-half the radius.

Let O be the centre of a circle whose radius is OA, and circumference $ABCD$, etc.; then will its area be equal to *circ.* $OA \times \frac{1}{2}\,OA$.



Inscribe in the circle a regular polygon $ABCD$, etc., and draw the radii OA, OB, etc., and the perpendicular OE. The area of each triangle of the polygon is equal to its base multiplied by one-half its altitude, and since the altitudes are equâl being radii of the inscribed circle, the area of the polygon is equal to the sum of the bases, or its perimeter multiplied by one-half of OE. Now, this is true whatever the number of sides; hence, it is true when the number of sides is infinite and the polygon becomes a circle. In this case the perimeter becomes the circumference, and the line $O\,E$, the radius. Therefore, the area of a circle is equal to the circumference multiplied by one-half of the radius.

Cor. The area of a circle is equal to the circumference multiplied by one-fourth of the diameter.

THEOREM XIV.

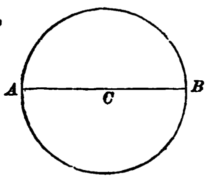
The area of a circle equals the square of the radius multiplied by π.

Let C be the centre of a circle; denote its radius CA by R, and its area by *area CA;* then from the previous theorem we have,

$$\textit{area } CA = \textit{circ. } CA \times \tfrac{1}{2} R;$$

but, $\textit{circ. } CA = 2\,\pi\,R$ (Th. XII. C.);

hence, $\textit{area } CA = 2\,\pi\,R \times \tfrac{1}{2} R,$



which, reduced, gives,

$$\textit{area } CA = \pi\,R^2.$$

Therefore, etc.

Cor. In a similar manner, we find that *area* $CA = \pi \frac{1}{4} D^2$, or *area* $CA = \frac{1}{4} \pi D^2$.

Scholium. The finding the exact length of the circumference of a circle is called the *rectification* of the circle. The finding of the area of a circle is called the *quadrature of* or *squaring the circle.* Both of these are celebrated problems, and can only be solved approximately, as may be shown by Calculus.

It was stated in Theorem XII. that the value of π is about 3.1416. This value is generally determined by finding a numerical expression for the area of a circle whose radius is unity, which area may be shown equal to the ratio of the circumference to the diameter. The solution is given in the following proposition.

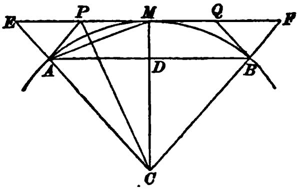
PROPOSITION XV.

PROBLEM.—*To find the numerical value of π, the ratio of the circumference to the diameter.*

The area of a circle equals πR^2; but when $R = 1$, the area of the circle equals π; hence, we may find the value of π by finding the area of a circle whose radius is 1. As a circle is a polygon of an infinite number of sides, by constructing successive similar inscribed and circumscribed polygons of double the number of sides, two may be found whose areas so nearly approach each other that either of them may be taken for the area of the circle.

Let C be the centre of the circle, AB the side of an inscribed, and EF of a circumscribed, polygon. Draw the chord AM, and the tangents AP and BQ; then AM will be the side of an inscribed, and PQ of a circumscribed poly-

gon of double the number of sides. Draw CE, CP, CM, and CF.



Let $P =$ area of given circumscribed polygon.

" $p =$ area of given inscribed polygon.

" $P' =$ area of circumscribed polygon of double the number of sides.

" $p' =$ area of inscribed polygon of double the number of sides.

1. The triangles CAD, CAM, and CEM are like parts of their equimultiples p, p', and P; hence, they are proportional to those polygons. But CAM is a mean proportional between CAD and CEM (B. III. Th. IX. C. 3); therefore, p' is a mean proportional between p and P, or

$$p' = \sqrt{p \times P}. \qquad (1)$$

2. In the triangles CAD and AEP, the angle $CAD = AEP$, EAP and ADC are right angles; hence, the triangles are similar, and

$$AC : CD :: EP : AP; \text{ but } AC = CM, \text{ and } AP = PM;$$

hence, $CM : CD :: EP : PM$.

Because of common altitude, CAM and CAD are to each other as CM to CD, and CEP to CPM as EP to PM.

Therefore, $CAM : CAD :: CEP : CPM$; by composition and multiplying consequents by 2,

$$CAM + CAD : 2\,CAD :: CEP + CPM : 2\,CPM,$$

p', p, P, and P', being respectively equimultiples of CAM, CAD, $CEP + CPM$, and $2\,CPM$;

we have, $$p' + p : 2p :: P : P' \therefore P' = \frac{2p \times P}{p' + p} \qquad (2)$$

Now, if p and P are squares, the area of the first is 2, and of the second is 4;

then, from (1), $p' = \sqrt{8} = 2.8284271$,

and, from (2), $P' = \dfrac{16}{2 + \sqrt{8}} = 3.3137085$;

which are the areas of the inscribed and circumscribed *octagons;* and in the same manner we may find the areas of polygons of 16, 32, etc. sides. For 8192 sides, the area of the inscribed polygon is 3.1415923 +, and of the circumscribed polygon, 3.1415928 +, either of which may be taken for the area of the circle whose radius is 1; and, since we have shown this to be the value of π, we have $\pi = 3.14159+$.

Scholium. The value of π is generally taken to be 3.1416.

NOTE.—We invite special attention to the method of treating the circumference and area of the circle, and also to the simple and concise method of presenting the derivation of the value of π, as given in the last proposition. The abbreviation in the method of deriving the *second formula* is due to Mr. Charles H. Harding, a former pupil and a present associate teacher.

PRACTICAL EXERCISES.

1. The radius of a circle is 6 inches; what is its circumference?

2. The diameter of a circle is 8 inches; what is its area?

3. The circumference of a circle is 50.2656 feet; required the radius.
Ans. 8 feet.

4. The area of a circle is 490.875 square inches; required the diameter and circumference.
Ans. Diameter, 25; circumference, 78.54.

5. The distance around a circular park is 180 rods; required the area of the park. *Ans.* 16 A. 18.23 P.

6. What is the length of an arc of 75° on the circumference of a circle whose radius is 5 feet? *Ans.* 6.545 feet.

7. How many degrees in an arc 18 inches long, on a circumference whose radius is 5 feet? *Ans.* 17° 11′ 19″.

8. A circle 20 feet in diameter is circumscribed by another circle 30 feet in diameter; what is the area of the space included between them?

9. A has a circular garden whose diameter is 18 rods, and B has one whose area is $2\frac{7}{9}$ times as great; what is the diameter of B's garden?

Ans. 30 rods.

10. Find the side of a square inscribed in a circle whose diameter is 5 feet. *Ans.* 3.535 feet.

11. Within a circular park 160 rods in circumference is a circular lake 80 rods in circumference; required the width of the ring of land surrounding the lake. *Ans.* 12.732 rods.

12. Deborah has a circular garden and John a square one, and the distance around each is 120 rods; which contains the most land, and how much? *Ans.* 245.95 square rods.

13. A man has a square garden and his wife a circular one, and each garden contains one acre; how much further around is one than the other? *Ans.* 5.756 rods.

14. The area of a circle is 314.16; if this circle be circumscribed by a square, required the area of the part between the circumference and the perimeter of the square. *Ans.* 85.84.

15. The area of a circle is 4 acres; required the side of the inscribed square, and the area of the part of the circle between the circumference and perimeter of the square. *Ans.* 1 A. 1 R. 32 P.

THEOREMS FOR ORIGINAL THOUGHT.

1. If two circumferences intersect, the distances between their centres will be less than the sum of their radii and greater than the difference.

2. If two circumferences intersect, the points of intersection will lie in a perpendicular to the line joining their centres, and at equal distances from it.

3. In equal circles the greater arc has the greater chord, and, conversely, the greater chord subtends the greater arc.

4. In equal circles, equal chords are equally distant from the centre, and the greater chord is nearer the centre.

5. If we inscribe a square in a circle, the radius is to the side of the inscribed square as 1 is to $\sqrt{2}$.

6. If a regular hexagon be inscribed in a circle, each side will be equal to the radius of the circle.

7. The area of a triangle is equal to the perimeter multiplied by one-half the radius of the inscribed circle.

8. In any inscribed quadrilateral, the sum of the opposite angles is equal to two right angles.

9. When a quadrilateral circumscribes a circle, the sums of its opposite sides are equal.

10. When the radius of a circle is unity, its area and semi-circumference are numerically equal.

PRACTICAL PROBLEMS IN GEOMETRICAL CONSTRUCTION,

INVOLVING THE PRINCIPLES OF BOOKS I., II., III., AND IV.

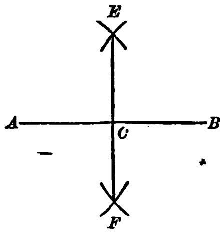
The following problems are solved by the principles of the previous books. The solution of a few is given in full; in others, the construction is given, and the reason for the solution indicated by referring to the theorem or theorems upon which it depends. The pupil will give the explanation in full.

The object of these is to teach the pupil to draw accurately upon paper. They are of great use in drawing the notes of a survey, or in representing any geometrical figure upon paper. The pupils need two instruments, a rule and compasses; with these all the following problems may be readily solved

PROBLEM I.

To bisect a given straight line.

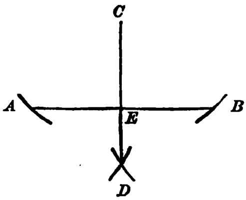
Let AB be the given straight line. From A and B, as centres, with a radius greater than one-half of AB, describe arcs intersecting at E and F; draw the line EF; then will C be the middle point of AB. For, E and F are each equally distant from A and B; hence, EC is perpendicular to AB (B. I. Th. XIV. C. 2); and from this it may be readily shown that $AC = CB$.



PROBLEM II.

From a given point without a straight line to draw a perpendicular to the line.

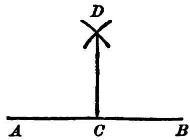
Let AB be the given line, and C the given point. From C as the centre, with a radius sufficiently great, describe an arc cutting the line AB in the two points A and B; then from A and B as centres, with a radius greater than one-half of AB, describe two arcs cutting each other in D, and draw CD; it will be the perpendicular required (B. I. Th. XIV. C. 2).



PROBLEM III.

At a given point in a straight line to erect a perpendicular to that line.

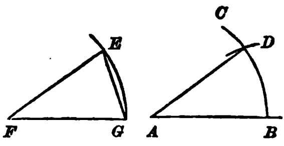
Let AB be the given line, and C the given point. Then, in the line AB take the points A and B equally distant from C, and with A and B as centres, and a radius greater than one-half of AB, describe two arcs cutting each other at D; draw DC; it will be the perpendicular required (B. I. Th. XIV. C. 2).



PROBLEM IV.

At a point on a given straight line to make an angle equal to a given angle.

Let A be the given point, AB the given line, and EFG the given angle.



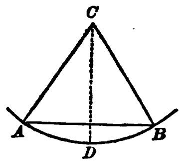
From the point F as a centre,

with any radius *FG*, describe the arc *EG*. From *A* as a centre, with the same radius, describe the arc *CB;* then, with a radius equal to the chord *EG*, describe an arc from *B* as a centre, cutting the arc *CB* in *D*, and draw *AD;* then will the angle *DAB* equal *EFG* (B. I. Th. IX.).

PROBLEM V.

To bisect a given arc, or a given angle.

First. Let *ADB* be the given arc, and *C* its centre. Draw the chord *AB*, and from *C* draw *CD* perpendicular to *AB* (P. II.); then will *CD* bisect *AB* (B. IV. Th. II.).

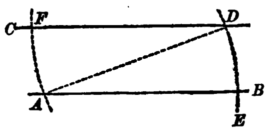


Second. Let *ACB* be the given angle. Then, with *C* as a centre and any radius *CA*, describe the arc *AB*, and bisect this arc by the line *CD*, as in the previous case; then will *CD* bisect *ACB* (B. IV. Th. II.).

PROBLEM VI.

Through a given point to draw a straight line parallel to a given straight line.

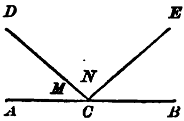
Let *A* be the given point and *CD* the given line. From *A* as a centre, with a radius greater than the shortest distance from *A* to *CD*, describe an indefinite arc *DE;* from *D* as a centre, with the same radius, describe the arc *AF;* take *DE* equal to *AF*, and draw *AB;* *AB* will be the parallel required.



For, drawing *AD*, we have *ADF* = *DAE* (B. IV. Th. VI.); hence, *AE* and *CD* are parallel (B. I. Th. IV.).

PROBLEM VII.

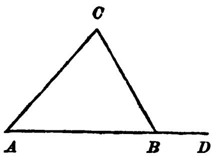
Two angles of a triangle being given, to find the third.



Let M and N be the given angles. Draw the indefinite line AB; at any point, as C, construct the angle ACD equal to M, and the angle DCE equal to N; then will ECB equal the third angle.

PROBLEM VIII.

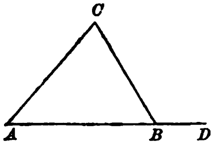
Given two sides and the included angle of a triangle, to construct the triangle.



Draw the indefinite line AD; take AB equal to one of the given sides; at A construct the angle A equal to the given angle, and take AC equal to the other given side; draw BC; then will ABC be the required triangle (B. I. Th. VI.).

PROBLEM IX.

Given one side and two angles of a triangle, to construct the triangle.



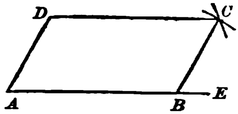
If the angles are not adjacent, find the third angle by P. VII.; we then have two angles and the included side, and proceed thus:—

Draw the indefinite line AD; take AB equal to the given side; at A make the angle BAC equal to one of the angles; at B make the angle ABC equal the other angle; then produce AC and BC till they meet, and ABC will be the required triangle (B. I. Th. VII.).

PROBLEM X.

Given two adjacent sides of a parallelogram and the included angle, to construct the parallelogram.

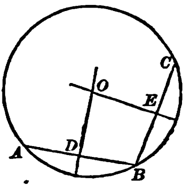
Draw the indefinite line AE, and upon it take AB equal to one of the sides. At A construct the angle BAD equal to the given angle, and take AD equal to the other given side. Draw DC parallel to AB, and BC parallel to AD; then will $ABCD$ be the parallelogram required (B. I. Th. XV. C. 2).



PROBLEM XI.

To find the centre of a given circumference or arc.

Take any three points, A, B, and C, on the circumference or arc, and unite them by the lines AB and BC. Bisect these chords by the perpendiculars DO and EO; then will their intersection O be the centre of the circle (B. IV. Th. III.).

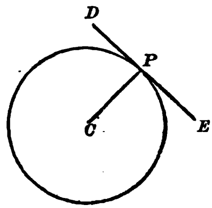


PROBLEM XII.

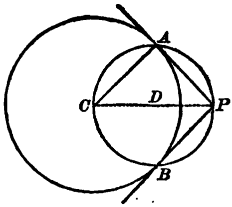
Through a given point to draw a tangent to a given circle.

First. Suppose the given point P to be in the circumference.

Find C, the centre of the circle (P. XI.); draw the radius CP; and then through P draw the perpendicular DE; DE will be the tangent required (B. IV. Th. IV.).



Second. Suppose the given point P



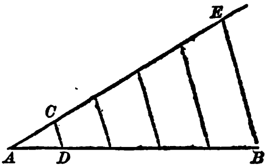
to be without the circle. Join P and the centre of the circle; bisect PC in D; with D as a centre, and a radius DC, describe the circumference intersecting the given circumference in A and B; draw PA or PB; then each of those will be the tangent required.

For, since CAP is a semicircle, the angle CAP is a right angle (B. IV. Th. VII. C. 1); hence, AP is a tangent (B. IV. Th. IV.).

PROBLEM XIII.

To divide a given line into any number of equal parts.

Let AB be the given line, and suppose we wish to divide it into any number, say 5 equal parts.



Through A draw the indefinite line AE, making any angle with AB. Take AC of any convenient length, and apply it 5 times to AE; join B with the last point of the division; and through the other points of division draw lines parallel to EB; then will AB be divided into 5 equal parts.

For, since DC and BE are parallel, we have (B. III. Th. IX.),

$$AC : AE :: AD : AB.$$

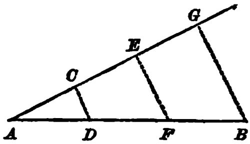
But AC is one-fifth part of AE; hence, AD is one-fifth part of AB.

PROBLEM XIV.

To divide a given line into parts proportional to given lines.

Let AB be the given line, to be divided into parts pro-

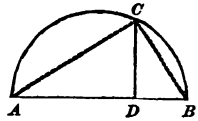
portional to the given lines P, Q, and R. Through A draw AG, making any angle with AB. On AG lay off AC equal P, CE equal Q, EG equal R; draw BG, and from the points C and E draw CD and EF parallel to GB; then will AD, DF, and FB be proportional to AC, CE, and EG (B. III Th. IX.).



PROBLEM XV.

To construct a mean proportional to two given lines.

Let P and Q be the two given lines. Draw an indefinite line, and on it lay off AD equal to P, and DB equal to Q; on AB as a diameter describe a semicircle, and draw DC perpendicular to AB; then, in the triangle ACB, will DC be a mean proportional to AD and DB (B. III. Th. XIV.).

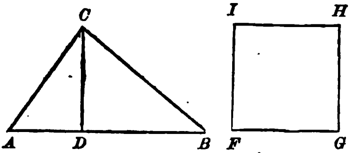


PROBLEM XVI.

To construct a square equal to a given triangle.

Let ABC be the given triangle, AB its base, and CD its altitude.

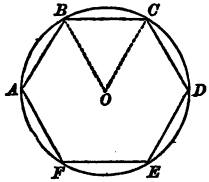
Find a mean proportional between CD and one-half of AB (Prob. XV.). Let FG be that mean proportional, and on it, as a side, construct the square $FGHI$; this will be the square required. For, by the construction, we have $\overline{FG}^2 = \frac{1}{2} AB \times CD$, which equals the area of ABC.



PROBLEM XVII.

To inscribe a regular hexagon in a circle.

Suppose the problem to be solved, and that $ABCDEF$ is a regular hexagon; draw the radii OB and OC. Now, the arc BC is one-sixth of a circumference, or 60°; hence, the angle BOC is 60°, and the other angles OBC and BCO equal 180° minus 60°, or 120°, and, OB being equal to OC, the angles OBC and BCO are equal; hence, each is equal to one-half of 120°, or 60°. Consequently, the triangle OBC is equiangular, and therefore equilateral; hence, the side BC is equal to the radius OB. Therefore, to inscribe a regular hexagon in a circle, we apply the radius six times as a chord to the circumference.



PROBLEMS FOR ORIGINAL THOUGHT.

1. Given the three sides of a triangle, to construct the triangle.

2. Given two sides of a triangle, and the angle opposite one of them, to construct the triangle.

3. To inscribe a circle in a given triangle.

4. To inscribe a circle in a square, and a square in a circle.

5. To find the side of a square which shall be equal to the sum of two given squares.

6. To find the side of a square which shall be equal to the difference between two given squares.

7. To construct a rectangle equal in area to a given triangle.

8. To find a fourth proportional to three given lines.

9. On a given line to construct a rectangle which shall be equal to a given rectangle.

10. To construct a square that shall be equal in area to a given parallelogram.

BOOK V.

PLANES AND THEIR ANGLES.

DEFINITIONS.

1. A **Plane** is a surface such that a straight line connecting any two of its points will lie entirely in the surface.

2. A straight line is PERPENDICULAR TO A PLANE when it is perpendicular to any line of the plane passing through its foot. The *foot* is the point where the line meets the plane.

Reciprocally, the plane is also perpendicular to the line.

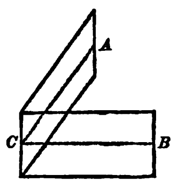
3. A straight line is PARALLEL TO A PLANE when it cannot meet the plane, however far both be produced.

Reciprocally, the plane is also parallel to the line.

4. TWO PLANES ARE PARALLEL when they cannot meet, however far both be produced.

5. When two planes meet, they form a line, which is called their LINE OF INTERSECTION.

6. A **Diedral Angle** is the divergence of two planes. The line in which the planes intersect is called the *edge of the angle;* the planes are called the *faces of the angle.*



A diedral angle is measured by the angle formed by two lines, one in each plane and perpendicular to the edge at the same point. Thus, the diedral angle in the margin is measured by the angle ACB.

7. A POLYEDRAL ANGLE is the divergence of three or more planes proceeding from a common point.

The common point is called the *vertex* of the angle; the planes are its *faces;* the intersection of the planes, its *edges*.

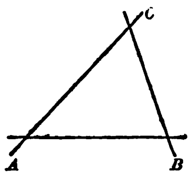
8. A TRIEDRAL ANGLE is a polyedral angle of three faces.

ANALYSIS.—This book treats of planes, the lines and angles formed by their intersection. It is not so valuable in itself as the other books of Geometry, and much less interesting. Its object is to prepare for the book which immediately follows it. The propositions are nearly self-evident, being apprehended almost immediately upon being heard. The demonstrations are simple and easily understood.

THEOREM I.

Through three points not in the same straight line, one plane can be passed, and but one.

Let A, B, and C be the three points; then can one plane be passed through them.



For, join two of the points, as A and C, by the line AC. Pass a plane through AC, and turn it around AC until it contains the point B; it will then pass through the three points A, C, and B. If now the plane be turned about AC, it will no longer contain the point B; hence, only this one plane can be passed through the three points. Therefore, etc.

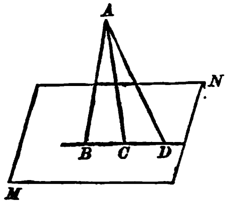
Cor. 1. Since only one plane can be passed through three points, three points are said to determine the position of a plane.

Cor. 2. Two lines which intersect, determine the position of a plane.

THEOREM II.

A perpendicular is the shortest line which can be drawn from a point to a plane.

Let A be a point without the plane MN; draw the perpendicular AB and oblique line AC; then will AB be shorter than AC.



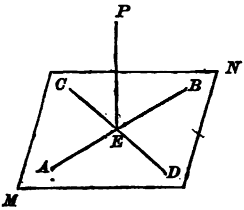
For, through B, the foot of the perpendicular, draw the line BC; then will AB be less than AC (B. I. Th. XIV.). Therefore, etc.

Cor. If several oblique lines, as AC, AD, be drawn, it is evident that those which meet the plane nearest B will be the shortest.

THEOREM III.

If a straight line is perpendicular to two straight lines of a plane at the point of their intersection, it is perpendicular to the plane of those lines.

Let the line PE be perpendicular to the two lines AB and CD of the plane MN at their point of intersection; then will it be perpendicular to the plane MN.

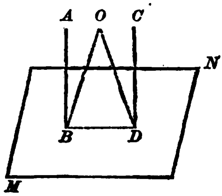


For, if PE is not perpendicular to MN, pass a plane through E which is perpendicular to PE. Then every line drawn through E, perpendicular to PE, will be a line of this last plane; but AB and CD are perpendicular to PE; hence, they are lines of this plane, and, therefore, PE is perpendicular to the plane of the lines AB and CD, which is the plane MN. Therefore, etc.

THEOREM IV.

If two straight lines are perpendicular to the same plane, they will be parallel to each other.

Let AB and CD be two straight lines perpendicular to the plane MN; then will they be parallel.



For, pass a plane through the lines, forming the section BD. Then the lines, being perpendicular to the plane, are perpendicular to the line BD; they are, therefore, parallel (B. I. Th. IV. C.).

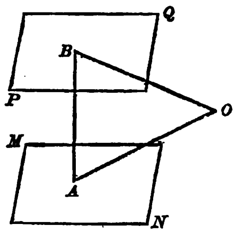
Cor. If one of two parallels is perpendicular to a plane, the other is also perpendicular to the same plane.

For, since the lines are parallel, they have the same direction; hence, if one is at right angles to the plane, the other must also be at right angles to the plane.

THEOREM V.

If two planes are perpendicular to the same straight line, they are parallel.

Let the two planes MN and PQ be perpendicular to the straight line AB; then will they be parallel.



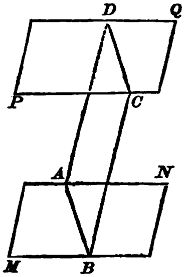
For, if they are not parallel, they will meet in some point O. From O draw the lines OA and OB; then, since OA lies in the plane MN, it will be perpendicular to AB at A (D. 2); and since OB lies in the plane PQ, it will be perpendicular to AB at B. Hence, we have two perpendiculars drawn from the same point to the same straight

line, which is impossible (B. I. Th. XIV. C. 1); consequently, the planes cannot meet, and are, therefore, parallel. Therefore, etc.

THEOREM VI.

If a plane meet two parallel planes, the lines of intersection are parallel.

Let the plane AC intersect the two parallel planes MN and PQ; then will AB and CD be parallel.



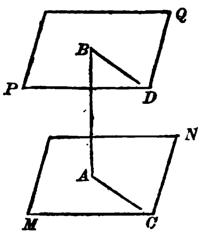
For, if the lines AB and CD are not parallel, since they lie in the same plane, they will meet if sufficiently produced, and, consequently, the planes MN and PQ will meet; but the planes cannot meet, since they are parallel; hence, the lines AB and CD cannot meet; they are, therefore, parallel.

Cor. Parallel lines included between parallel planes are equal. For, the opposite sides of the figure AC being parallel, it is a parallelogram, and hence AD equals BC.

THEOREM VII.

If a straight line is perpendicular to one of two parallel planes, it is perpendicular to the other also.

Let MN and PQ be two parallel planes, and let the line AB be perpendicular to PQ; then will it also be perpendicular to the plane MN.



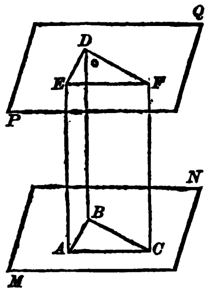
For, pass any plane through AB; the intersections AC and BD will be parallel (Th. VI.); since AB is perpendicular to PQ, it will be perpendicular to BD (D. 2), and since BD and AC are parallel,

it will be perpendicular to AC (B. I. Th. III. C.); hence, BA, being perpendicular to any line of the plane MN passing through its foot, is perpendicular to the plane MN. Therefore, etc.

THEOREM VIII.

If two angles not in the same plane have their sides parallel and lying in the same direction, the angles will be equal and their planes parallel.

Let BAC and DEF be two angles not in the same plane, having their sides respectively parallel and lying in the same direction; then will these angles be equal and their planes parallel.



Take ED equal to AB, and EF equal to AC, and draw BC, DF, AE, BD, and CF.

First. The angles BAC and DEF will be equal.

For, since AC and EF are equal and parallel, the figure $ACFE$ is a parallelogram (B. I. Th. XVII.), and AE and CF are equal and parallel. Since AB and ED are equal and parallel, $ABDE$ is a parallelogram, and AE and BD are equal and parallel; hence, BD and CF are equal and parallel, and, consequently, DF is equal and parallel to BC. Hence, the triangles ABC and EDF have their corresponding sides equal; they are, therefore, equal, and the angle DEF equals the angle BAC.

Second. The planes are parallel.

For, three lines which intersect determine the position of a plane; and since the three sides of the triangles are respectively parallel, their planes must be parallel.

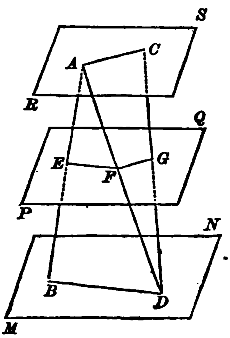
Cor. If three straight lines not in the same plane are equal and parallel, the triangles formed by joining the extremities of these lines will be equal, and their planes parallel. This is readily proven; let the pupil show it.

THEOREM IX.

If two straight lines are cut by three parallel planes, they will be divided proportionally.

Let the lines AB and CD be cut by the parallel planes MN, PQ, and RS, in the points A, E, B, and C, G, D; then will

$$AE : EB :: CG : GD.$$



For, draw the line AD, meeting the plane PQ in F; draw also AC, EF, FG, and BD. Now, since the planes MN and PQ are parallel, EF is parallel to BD (Th. VI.); and since PQ and RS are parallel, AC is parallel to FG. Hence (B. III. Th. IX.), we have,

$$AE : EB :: AF : FD; \text{ and also,}$$
$$AF : FD :: CG : GD.$$

Hence, from the principles of proportion, we have,

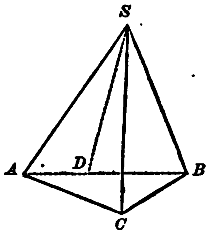
$$AE : EB :: CG : GD.$$

Therefore, etc.

THEOREM X.

Either angle of the three plane angles which form a triedral angle, is less than the sum of the other two.

Let the triedral angle whose vertex is S be formed by the three plane angles ASC, ASB, and CSB; then will any one of these be less than the sum of the other two.



If the angle considered is less than either of the other two, it is evidently less than their sum. Suppose, however, the angle greater than either of the other two, and let ASB be that angle. In the plane ASB make the angle BSD equal to BSC, draw the line AB at pleasure, make SC equal to SD, and draw AC and BC.

In the two triangles BSC and BSD, BS is common, CS equals DS, and the angle BSC equals BSD by construction; hence, the triangles are equal, and BD equals BC. Now (B. I. A. 10, C.),

$$AD + DB < AC + BC.$$

And, taking away the equals DB and BC, we have,

$$AD < AC.$$

Hence (B. I. Th. VIII.), we have,

$$\text{angle } ASD < \text{angle } ASC;$$

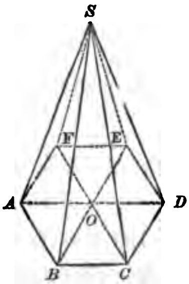
and, adding the equal angles DSB and CSB, we have,

$$\text{angle } ASB < \text{angle } ASC + \text{angle } CSB.$$

Therefore, etc.

THEOREM XI.

The sum of the plane angles which form any polyedral angle, is less than four right angles.



Let S be the vertex of a polyegral angle formed by the plane angles ASB, BSC, CSD, etc.; then will the sum of these plane angles be less than four right angles.

For, pass a plane cutting the edges in the points A, B, C, D, E, and F, and the faces in the lines AB, BC, etc. From any point, O, in the polygon thus formed,

draw the lines OA, OB, OC, etc. We then have two sets of triangles, one set having their vertices at S, the other at O, and both having the common bases AB, BC, etc.

Now, the sum of the angles of the upper set of triangles is equal to the sum of the angles of the lower set of triangles, since both sets consist of the same number of triangles. But the sum of the angles SBA and SBC is greater than ABC, or $ABO + OBC$ (Th. X.); and also $SCB + SCD$ is greater than $OCB + OCD$; and so on with the other angles at D, E, etc. Hence, the sum of all the angles at the bases of the upper set of triangles is greater than the sum of all the angles at the bases of the lower set of triangles; therefore, the sum of the angles at S must be less than the sum of the angles at O. But the sum of the angles at O is equal to four right angles (B. I. Th. II. C. 2); hence, the sum of the angles at S is less than four right angles. Therefore, etc.

Scholium. This proposition supposes that the polyedral angle is convex; if it were not, the sum of the plane angles would be unlimited.

THEOREMS FOR ORIGINAL THOUGHT.

1. Prove that but one plane can be passed through a given point perpendicular to a given line.

2. If a line is perpendicular to a plane, every plane passed through the line is perpendicular to that plane.

3. If two planes are perpendicular to each other, a line drawn in one of them perpendicular to their intersection is perpendicular to the other.

4. If two planes which cut each other are both perpendicular to a third plane, their intersection is perpendicular to that plane.

5. Prove that through a given line of a given plane, only one plane perpendicular to the given plane can be passed.

6. Prove that through a line parallel to a given plane, only one plane perpendicular to the given plane can be passed.

7. If two planes which intersect contain two lines parallel to each other, the intersection of the planes will be parallel to the lines.

8. If a line is parallel to one plane and perpendicular to another, these two planes are perpendicular.

9. Only one plane can be drawn through a given point parallel to a given plane.

10. If two planes are parallel to a third, they are parallel to each other.

11. If two lines are parallel in space, and planes be passed through them perpendicular to a third plane, the two planes will be parallel.

PROBLEMS.

The following problems are easily solved from the principles already presented.

1. To erect a perpendicular to a given plane at a given point of the plane. (See Prop. III.)

2. To construct a plane parallel to a given plane.

3. To construct a plane perpendicular to a given plane intersecting it in a given straight line.

4. To draw a line from a given point of a plane making any given angle with the plane.

5. To draw a plane intersecting a given plane and making any given angle with it.

BOOK VI.

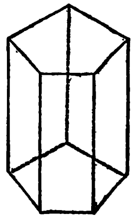
POLYEDRONS.

DEFINITIONS.

1. A **Polyedron** is a volume bounded by polygons.

The bounding polygons are called the *faces* of the polyedron; the lines in which the faces meet are called *edges;* and the points in which the edges meet are called *vertices* of the polyedron.

2. A **Prism** is a polyedron, two of whose faces are equal polygons, having their homologous sides parallel; the other faces are parallelograms.



The equal polygons are called *bases* of the prism; one, the *upper base;* the other, the *lower base.* The parallelograms constitute the *lateral* or *convex surface* of the prism; the intersections of the lateral faces are called *lateral edges.*

3. The **Altitude** of a prism is the perpendicular distance between its bases.

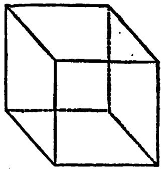
4. A **Right Prism** is one whose lateral edges are perpendicular to the bases. In a right prism, each lateral edge is equal to the altitude.

5. An **Oblique Prism** is one whose lateral edges are oblique to the bases. Each edge is, consequently, greater than the altitude.

6. A prism is named from its bases. A *triangular prism* is one whose bases are triangles; a *quadrangular prism* is one whose bases are *quadrilaterals;* and so on.

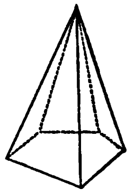
7. A PARALLELOPIPEDON is a prism whose bases are parallelograms.

A RECTANGULAR PARALLELOPIPEDON is a *right* parallelopipedon with rectangular bases. A *cube* is a rectangular parallelopipedon, all of whose faces are equal squares.



8. A PYRAMID is a polyedron bounded by a polygon, and by triangles meeting at a common point.

The polygon is called the *base* of the pyramid; the triangles, its *lateral* or *convex surface;* and the point where the triangles meet, its *vertex.*



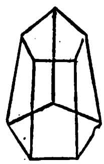
9. Pyramids are named from their bases; thus, we have *triangular*, *quadrangular*, *pentangular*, etc. pyramids, as the bases are triangles, quadrilaterals, pentagons, etc.

10. The ALTITUDE of a pyramid is the perpendicular distance from the vertex to the plane of the base.

11. A RIGHT PYRAMID is one whose base is a regular polygon, and in which a perpendicular from the vertex to the base passes through the centre of the base. This perpendicular is called the axis of the pyramid.

12. The SLANT HEIGHT of a right pyramid is the perpendicular distance from its vertex to any side of the base.

13. A FRUSTUM OF A PYRAMID is the part of a pyramid included between its base and a plane cutting the pyramid parallel to the base.



14. The ALTITUDE of a frustum of a pyramid is the perpendicular distance between its bases.

15. The SLANT HEIGHT of a frustum of a

right pyramid is that portion of the slant height of the pyramid included between the bases of the frustum.

16. SIMILAR POLYEDRONS are those which are bounded by the same number of similar polygons, similarly placed. Parts which are similarly placed are *homologous*, whether faces, angles, or edges.

17. The DIAGONAL of a polyedron is a line joining the vertices of any two polyedral angles not in the same face.

18. The VOLUME of a polyedron is its numerical value, expressing how many times it contains some other polyedron as a unit.

ANALYSIS.—This book treats of prisms, pyramids, and frustums. The object is to find the surface and volume of these polyedrons, and the relation of those which are similar. Their surface is readily determined by finding the area of the polygons which form their faces. In finding their volumes, we begin with the rectangular parallelopipedon, assuming for a *unit of measure* a cube whose edge is a unit of measure of the edges of the parallelopipedon. From the volume of a rectangular parallelopipedon we pass to that of any parallelopipedon, thence to the volume of a triangular prism, and from this to that of any prism. The division of a triangular prism into three equal parts gives the volume of a triangular pyramid, from which we pass to the volume of any pyramid, and also of any frustum.

In the method of treatment, we have made frequent use of the doctrine of infinites, by which many of the demonstrations are simplified, and several propositions usually included in this book are omitted.

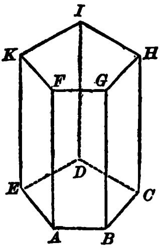
THE PRISM.

THEOREM I.

The convex surface of a right prism is equal to the perimeter of the base multiplied by the altitude.

Let $ABCDE—K$ be a right prism; then will its convex surface be equal to

$(AB+BC+CD+DE+EA)\times AF.$



For, the convex surface of the prism is equal to the sum of all the rectangles AG, BH, CI, etc. Now, the altitude of each of these rectangles is equal to the altitude of the prism, and the area of each rectangle is equal to its base multiplied by its altitude; hence, the convex surface, which is the sum of the areas of these rectangles, is equal to

$$(AB+BC+CD+DE+EA)\times AF;$$

or, the perimeter of the base multiplied by the altitude. Therefore, etc.

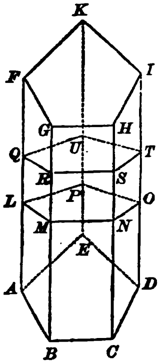
Cor. If two right prisms have the same altitude, their convex surfaces are to each other as the perimeters of their bases.

THEOREM II.

If a prism be cut by parallel planes, the sections formed will be equal polygons.

Let the prism $ABCDE—K$ be cut by the parallel planes

LO and QT; then will the sections LO and QT be equal polygons.



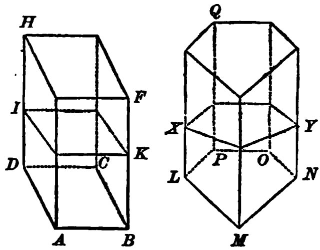
For, LM and QR are parallel, being the intersections of the two parallel planes with $ABGF$ (B. V. Th. VI.); these lines LM and QR are also equal, since they are parallels included between the two parallels AF and BG (B. I. Th. XIV. C. 2.). For a like reason, MN is equal and parallel to RS, NO to ST, OP to TU, etc.; hence, the angle LMN is equal to the angle QRS, MNO to RST, etc. (B. V. Th. VIII.); therefore, the sections are equal polygons.

Cor. Every section of a prism parallel to the base is equal to the base.

THEOREM III.

Any two prisms having equal bases and equal altitudes are equal in volume.

Let $ABCD—H$ and $LMNOP—Q$ be two prisms, having equal bases AC and LN, and equal altitudes; then will they be equal in volume.



For, let IK and XY be any two sections parallel to the bases, and at equal distances from them. Then, by the preceding theorem, the section IK will be equal to the base AC, and the section XY will be equal to the base LN; but the bases AC and LN are equal, by hypothesis; hence, the sections IK and XY are also equal; and in the same

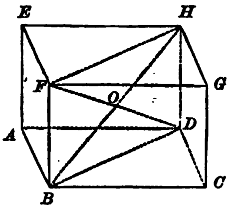
manner it may be shown that any other corresponding sections are equal. Since, then, any section of the prism $ABCD—H$ is equal to the corresponding section of the prism $LMNOP—Q$, the two prisms which are composed of these equal sections must be equal in volume. Therefore, etc.

Cor. Any prism is equal in volume to a rectangular parallelopipedon having an equal base and the same altitude.

THEOREM IV.

The opposite faces of a parallelopipedon are equal and parallel.

Let $ABCD—H$ be a parallelopipedon; then will its opposite faces be equal and parallel.



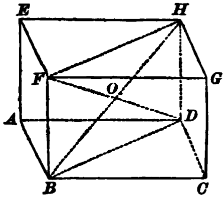
For, the bases are equal and parallel, by the definition of a parallelopipedon. Also, BC is equal and parallel to AD, since the figure $ABCD$ is a parallelogram, and, for a similar reason, BF and AE are equal and parallel; consequently, the angles EAD and FBC are equal, and their planes parallel (B. V. Th. VIII.), and, therefore, the parallelograms BG and AH are equal (B. I. Th. XV. C. 3). In a similar manner it may be shown that the faces AF and DG are equal and parallel. Therefore, etc.

Cor. 1. Any two opposite faces of a parallelopipedon may be taken as the bases.

Cor. 2. *The diagonals of a parallelopipedon bisect each other.*

Draw the diagonals FD and BH; draw also BD and FH; then, since BF and HD are equal and parallel, the

figure $BDHF$ is a parallelogram; hence, the diagonals FD and BH bisect each other at O (B. I. Th. XVIII.). In the same manner it may be shown that either of these and any other diagonal bisect each other; hence, all the diagonals bisect each other.



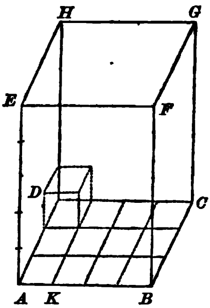
Cor. 3. In a rectangular parallelopipedon, the square of either diagonal equals the sum of the squares of the three edges which meet at the same vertex. Let the pupil show it; that $\overline{BH}^2 = \overline{BC}^2 + \overline{DC}^2 + \overline{DH}^2$.

THEOREM V.

The volume of a rectangular parallelopipedon is equal to the product of its base and altitude.

Let $ABCD—H$ be a rectangular parallelopipedon; then will its volume be equal to its base $ABCD$ multiplied by its altitude AE.

Suppose AK to be a common unit of measure of the three sides AB, AD, and AE, and suppose it to be contained 4 times in AB, 3 times in AD, and 5 times in AE; then divide AB into 4 equal parts, AD into 3, and AE into 5 equal parts, and pass planes through the points of division parallel to the faces of the parallelopipedon. The parallelopipedon will thus be divided into equal cubes, equal since their sides are equal and their angles are equal, all being right angles.



Now, the number of these little cubes upon the base is

equal to the number of surface units in the base, and the whole number of cubes in the parallelopipedon is equal to the number upon the base multiplied by the number of layers, and the number of layers is the same as the number of units in the altitude; hence, the number of cubic units in the parallelopipedon is equal to the base multiplied by the altitude. Now, this is evidently true whatever be the size of the linear unit; hence, it is true when the linear unit is exceedingly small, and, consequently, when it is infinitely small, as it must be when the three sides are incommensurable. Therefore, the volume of a rectangular parallelopipedon is equal to the product of its base and altitude.

Cor. 1. It is evident that the number of cubic units upon the base is equal to the number of rows multiplied by the number in each row; that is, the length of the base multiplied by its breadth; hence, *the volume of a rectangular parallelopipedon equals the product of its length, breadth, and altitude, or the product of its three dimensions.*

Cor. 2. Any two rectangular parallelopipedons are to each other as the product of their bases and altitudes, or as the product of their three dimensions.

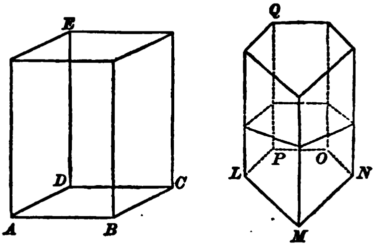
Cor. 3. When their bases are equal, they are to each other as their altitudes; when their altitudes are equal, they are to each other as their bases.

THEOREM VI.

The volume of any parallelopipedon, and in general of any prism, is equal to the product of its base and altitude.

Let *ABCD—E* be any parallelopipedon, and *LMNOP—Q* be any prism; then will the volume of each be equal to

the product of its base and altitude.



For, each of these volumes is equal to a rectangular parallelopipedon having an equal base and the same altitude (Th. III.). But the volume of a rectangular parallelopipedon is equal to the product of its base and altitude (Th. V.); therefore, the volume of each of these prisms is equal to the product of its base by its altitude. Therefore, etc.

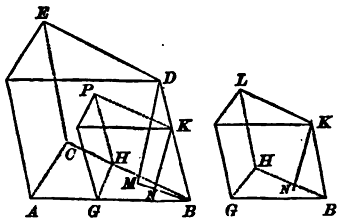
Cor. 1. Any two prisms are to each other as the products of their bases and altitudes.

Cor. 2. Prisms having equal bases are to each other as their altitudes; prisms having equal altitudes are to each other as their bases.

THEOREM VII.

Similar triangular prisms are to each other as the cubes of their homologous edges.

Let $ABC—E$ and $GBH—L$ be two similar triangular prisms; then will they be to each other as the cube of any two homologous edges AB and GB.



For, since the two prisms are similar, the faces containing the triedral angles B and B are respectively similar; therefore, the prism $GBH—L$ being applied to the prism $ABC—E$ will take the position $GBH—P$. From

D draw DM perpendicular to the base, and from K draw KN perpendicular to the base; then the two triangles DMB and KNB must be similar, since they are mutually equiangular.

Now, since the bases are similar, we have (B. III. Th. XVI.),

$$base\ ABC : base\ GBH :: \overline{AB}^2 : \overline{GB}^2;$$

and, since the triangles DMB and KNB are similar, and also the parallelograms AD and GK, we have,

$$DM : KN :: DB : KB :: AB : GB.$$

Multiplying together the corresponding terms of the first and last couplets of these two proportions, we have,

$$base\ ABC \times DM : base\ GBH \times KN :: \overline{AB}^3 : \overline{GB}^3.$$

But $base\ ABC \times DM$ is the volume of the prism $ABC—E$, and $base\ GBH \times KN$ is the volume of the prism $GBH—L$; hence, the prisms are to each other as $\overline{AB}^3$ to $\overline{GB}^3$. Therefore, etc.

Cor. 1. *Any two similar prisms are to each other as the cubes of their homologous edges.*

For, since the prisms are similar, their bases are similar, and may, therefore, be divided into the same number of similar triangles, similarly situated (B. III. Th. XVII.); hence, each prism may be divided into the same number of similar triangular prisms. But these triangular prisms are to each other as the cubes of their homologous edges; hence, the polygonal prisms which are the sum of these triangular prisms must be to each other as the cubes of their homologous edges.

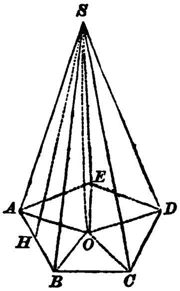
Cor. 2. Similar prisms are to each other as the cubes of their altitudes, or as the cubes of any other homologous lines.

THE PYRAMID.

THEOREM VIII.

The convex surface of a right pyramid is equal to the perimeter of the base multiplied by one-half of the slant height.

Let $ABCDE—S$ be a right pyramid, and SH the slant height; then will the convex surface be equal to the perimeter $AB + BC + CD + DE + EA$ multiplied by $\frac{1}{2}$ of SH.



Draw SO perpendicular to the base; then, from the definition of a right pyramid, O is the centre of the base; consequently, the distances AO, BO, CO, etc. are all equal, and therefore the edges SA, SB, SC, etc. are all equal (B. V. Th. II. C. 1); and, since the sides AB, BC, etc. are all equal, the triangles SAB, SBC, etc. are all equal, and their altitudes, which is the slant height of the pyramid, are equal.

Now, the area of each triangle is equal to its base multiplied by one-half of its altitude; hence, the sum of the areas of these triangles, which is the convex surface of the pyramid, equals the sum of their bases into one-half of the slant height SH; that is, the convex surface of the pyramid equals

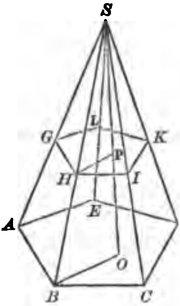
$$(AB + BC + CD + DE + EA) \times \tfrac{1}{2} SH.$$

Therefore, etc.

THEOREM IX.

If a pyramid be cut by a plane parallel to the base;
1. *The edges and altitude will be divided proportionally.*
2. *The section will be a polygon similar to the base.*

Let the pyramid $S—ABCDE$ be cut by a plane $GHIKL$ parallel to the base; then will the edges SA, SB, SC, etc., with the altitude SO, be divided proportionally, and the section $GHIKL$ will be similar to the base.



First. Since the planes $ABCDE$ and $GHIKL$ are parallel, the intersections AB and GH are parallel (B. V. Th. VI.); for the same reason, BC is parallel to HI, and BO to HP. Hence, we have (B. III. Th. IX. C. 1),

$$SA : SG :: SB : SH;$$

and also, $$SB : SH :: SC : SI;$$

and also, $$SB : SH :: SO : SP.$$

Hence, the edges and altitude are divided proportionally.

Second. Since GH is parallel to AB, and HI to BC, the angle GHI is equal to ABC (B. V. Th. VIII.); and, for the same reason, each angle of the polygon $GHIKL$ is equal to the corresponding angle of the base; hence, the two polygons are mutually equiangular.

Again, since GH is parallel to AB, we have,

$$GH : AB :: SH : SB;$$

and, since HI is parallel to BC, we have,

$$HI : BC :: SH : SB.$$

Hence, from equal ratios, we have,

$$GH : AB :: HI : BC.$$

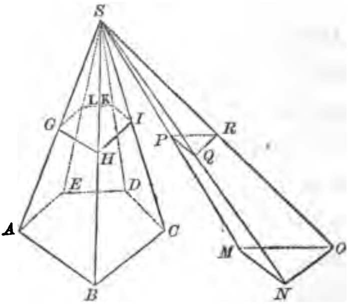
In the same manner, it may be shown that all the sides

of the two polygons are proportional; hence, the section $GHIKL$ is similar to the base $ABCDE$ (B. III. D. 6).

THEOREM X.

If two pyramids have the same altitude, and their bases in the same plane, the sections made by a plane parallel to their bases are to each other as their bases.

Let $S—ABCDE$ and $S—MNO$ be two pyramids, having the same altitude, and their bases in the same plane; and let $GHIKL$ and PQR be sections made by a plane parallel to their bases; then will these sections be to each other as the bases.



For, the polygons $ABCDE$ and $GHIKL$, being similar, are to each other as the squares of their sides AB and GH (B. III. Th. XVIII.); but

$$AB : GH :: SA : SG.$$

Hence, $$ABCDE : GHIKL :: \overline{SA}^2 : \overline{SG}^2.$$

For a similar reason,

$$MNO : PQR :: \overline{SM}^2 : \overline{SP}^2.$$

But (B. V. Th. IX.) we have,

$$SA : SG :: SM : SP :$$

Hence, $$ABCDE : GHIKL :: MNO : PQR.$$

Therefore, etc.

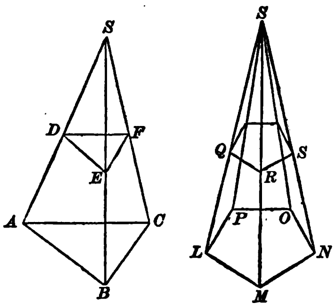
Cor. 1. If the bases are equal, any sections at equal distances from the bases are equal.

Cor. 2. Any section parallel to the base is proportional to the square of its distance from the vertex.

THEOREM XI.

Pyramids having equal bases and the same altitude are equal in volume.

Let $S—ABC$ and $S'—LMNOP$ be two pyramids, having equal bases and equal altitudes; then will they be equal in volume.



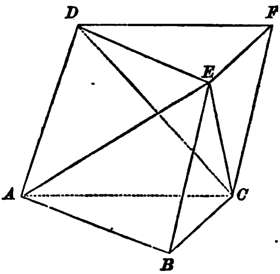
For, pass a plane parallel to their bases ABC and $LMNOP$, at equal distances from their bases; then these sections are equal (Th. X. C. 1); and in the same manner it may be shown that any other sections at equal distances from the bases are equal.

Since, then, every section in one pyramid is equal to a corresponding section in the other, and the altitudes are equal, the pyramids composed of these equal sections are equal in volume.

THEOREM XII.

A triangular prism may be divided into three equal triangular pyramids.

Let $ABC—F$ be a triangular prism; then may it be divided into three equal triangular pyramids.



Pass a plane through the edge AC and the point E, cutting off the pyramid $ABC—E$; pass another plane through DE and the

point C, cutting off the pyramid $DEF—C$; there will remain a pyramid whose base may be regarded as ACD, having its vertex at E. Now, the two pyramids $ABC—E$ and $DEF—C$ are equal in volume, since they have equal bases and equal altitudes (Th. XI.). Regarding the pyramid $DEF—C$ as having the base DCF and vertex at E, it is equal in volume to the pyramid $ACD—E$, since their bases are equal, being halves of the parallelogram $ACFD$, and their altitudes are equal, since their bases are in the same plane and vertices at the same point. Hence, the three pyramids into which the prism is divided are all equal in volume. Therefore, etc.

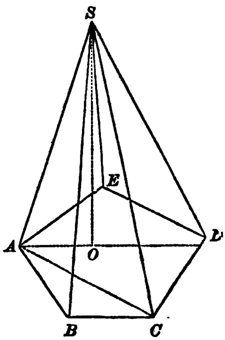
Cor. 1. A triangular pyramid is one-third of a prism having an equal base and an equal altitude.

Cor. 2. The volume of a triangular pyramid is one-third of the product of its base and altitude.

THEOREM XIII.

The volume of a pyramid is equal to one-third of the product of its base and altitude.

Let $S—ABCDE$ be a pyramid, and SO the altitude; then will its volume be equal to $ABCDE \times \frac{1}{3} SO$.



Draw the diagonals AC and AD, and pass the planes SAC and SAD through these diagonals and the vertex S; the pyramid will then be divided into triangular pyramids, whose altitudes are equal, being the altitude of the pyramid. Now, the volume of each of these triangular pyramids is equal to its base by one-third of the altitude

(Th. XII. C. 2); hence, the volume of the pyramid *S—ABCDE*, which is the sum of these triangular pyramids, is equal to the sum of their bases into one-third of the altitude; that is, base $ABCDE \times \frac{1}{3} SO$. Therefore, etc.

Cor. 1. The volume of a pyramid is one-third of the volume of a prism having an equal base and an equal altitude.

Cor. 2. Pyramids are to each other as the products of their bases and altitudes.

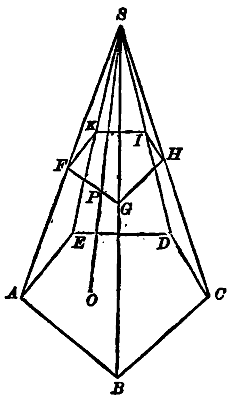
Cor. 3. Pyramids having equal bases are to each other as their altitudes; pyramids having equal altitudes are to each other as their bases.

Scholium. The volume of any polyedron may be found by dividing it into triangular pyramids, by passing planes through its vertices.

THEOREM XIV.

Similar pyramids are to each other as the cubes of their homologous edges.

Let *S—ABCDE* and *S—FGHIK* be two similar pyramids; then will they be to each other as the cubes of any two homologous sides *AB* and *FG*.



For, since the pyramids are similar, they may be so placed that their homologous angles at the vertex will coincide. Then, since the faces *SAB* and *SFG* are similar, *AB* is parallel to *FG*; and since *SBC* and *SGH* are similar, *BC* is parallel to *GH*; hence, the planes of the bases are parallel (B. V. Th. VIII.).

Draw SO perpendicular to the base $ABCDE$; it will also be perpendicular to the base $FGHIK$ at some point, P; then (Th. IX.),

$$SO : SP :: SB : SG :: AB : FG;$$

and, consequently,

$$\tfrac{1}{3}\, SO : \tfrac{1}{3}\, SP :: AB : FG.$$

But, the bases of the pyramids being similar, we have (B. III. Th. XVIII.),

$$base\ ABCDE : base\ FGHIK :: \overline{AB}^2 : \overline{FG}^2.$$

Multiplying these two proportions, term by term, we have,

$$base\ ABCDE \times \tfrac{1}{3}\, SO : base\ FGHIK \times \tfrac{1}{3}\, SP :: \overline{AB}^3 : \overline{FG}^3.$$

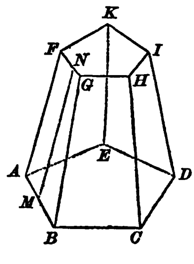
But, *base* $ABCDE \times \frac{1}{3}\, SO$ is equal to the volume of the pyramid S—$ABCDE$, and *base* $FGHIK \times \frac{1}{3}\, SP$ is equal to the volume of the pyramid S—$FGHIK$; hence, the two pyramids are to each other as the cubes of the homologous edges AB and FG. Therefore, etc.

Cor. Similar pyramids are to each other as the cubes of their altitudes, or as the cubes of any two homologous lines.

FRUSTUM OF A PYRAMID.

THEOREM XV.

The convex surface of a frustum of a right pyramid is equal to one-half of the sum of the perimeters of the upper and lower bases, multiplied by the slant height.



Let $ABCDE$—K be the frustum of a right pyramid, and NM its slant height; then will its convex surface be equal to one-half of the sum of the perimeters of its two bases, multiplied by NM.

The faces forming the convex surface are equal trapezoids; for the faces of the

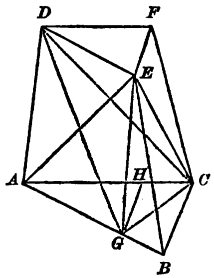
pyramid of which this frustum is a part are equal, and the faces of the pyramid cut off are equal; hence, the figures which remain are equal, and their upper and lower bases being parallel, they are equal trapezoids, and have a common altitude NM, the slant height of the frustum.

Now, the area of each trapezoid, as $ABGF$, is equal to $\frac{1}{2}(AB + FG) \times NM$ (B. III. Th. IV.); hence, the area of the convex surface, which is the sum of all the trapezoids, is equal to one-half the sum of the perimeters of the upper and lower bases multiplied by the slant height. Therefore, etc.

THEOREM XVI.

The volume of a frustum of a triangular pyramid is equal to the sum of the volumes of three pyramids, whose common altitude is the altitude of the frustum, and whose bases are the lower base of the frustum, the upper base of the frustum, and a mean proportional between the two bases.

Let $ABC—F$ be the frustum of a triangular pyramid. Through the points A, E, C, pass a plane cutting off the pyramid $E—ABC$. This pyramid has the altitude of the frustum, and for its base the lower base of the frustum. Through the points D, E, C, pass a plane cutting off the pyramid $C—DEF$. This pyramid has the altitude of the frustum, and for its base the upper base of the frustum. The remaining part of the frustum is a pyramid whose base is ACD, with its vertex at E.



Now, draw EG parallel to DA; draw also GD; then the pyramid $E—ACD$ is equal to the pyramid $G—ACD$, since they have the same base and altitude. But the pyramid

G—ACD may be regarded as having *AGC* for its base, and its vertex at *D;* it will then have the altitude of the frustum. We will now show that its base *AGC* is a mean proportional between the two bases of the frustum.

Draw *GH* parallel to *BC;* then the triangles *AGH* and *DEF*, being similar to *ABC*, are similar to each other, and, hence, equiangular; and since *AG* equals *DE*, the triangle *AGH* equals *DEF* (B. I. Th. VII.). Now, *AGC* is a mean proportional between *AGH* and *ABC* (B. III. Th. IX. C. 3); hence, the base of the third pyramid is a mean proportional between the upper and lower bases. Therefore, etc.

Cor. This proposition is true for the frustum of any pyramid. For, since any pyramid is equal to a triangular pyramid having an equal base and equal altitude, by cutting the pyramids with a plane parallel to the base, and removing the upper part, it may be shown that the frustum of any pyramid is equal to the frustum of a triangular pyramid having equal bases and the same altitude; hence, if the proposition is true for triangular frustums, it is true for all frustums.

PRACTICAL EXAMPLES.

1. Required the convex surface of a right prism whose altitude is 14 inches and perimeter of the base 16 inches. *Ans.* 224 square inches.

2. Required the contents of a prism the area of whose base is 24 square feet and altitude 7 feet. *Ans.* 168 cubic feet.

3. Required the convex surface of a regular pentangular pyramid whose slant height is 18 inches and each side of the base 6 inches. *Ans.* 270 square inches.

4. Required the volume of the frustum of a square pyramid, the sides of whose bases are 8 and 6 inches, and whose altitude is 12 inches. *Ans.* 592 cubic inches.

5. Required the entire surface of a cube whose sides are each 11 inches. *Ans.* 726 square inches.

6. A man wishes to make a cubical cistern whose contents are 373248 cubic inches; how many feet of inch boards will line it?

Ans. 180 square feet.

7. What is the side of a cube which contains as much as a volume 20 feet 6 inches long, 10 feet 8 inches wide, and 6 feet 9 inches high?

Ans. 11.4 feet.

8. What is the depth of a cubical cistern which shall contain 1600 gallons, each 231 cubic inches of water? *Ans.* 5.98 feet.

9. Required the dimensions of a cube whose surface shall be numerically equal to its contents. *Ans.* 6 units.

10. There are two similar prisms whose lengths are as 7 to 28 respectively; required the relation of their contents. *Ans.* 1 : 64.

11. Required the contents of a pyramid whose altitude is 20 inches and whose base is a regular hexagon, each side being 6 inches.

Ans. 623.5386 cubic inches.

12. If we pass a plane parallel to the base of the pyramid of the 11th problem, half-way between its vertex and base, required the convex surface and contents of the frustum.

Ans. Vol. = 545.596 cubic inches.

13. A farmer wishes to know what must be the depth of a cubical box which shall contain 100 bushels of grain, each bushel 2150.42 cubic inches. *Ans.* 4.9 feet.

THEOREMS FOR ORIGINAL THOUGHT.

1. Parallelopipedons having equal bases and equal altitudes are equal in volume.

2. The diagonals of a rectangular parallelopipedon are equal.

3. If a plane be passed through the opposite edges of a rectangular parallelopipedon, the triangular prisms formed are equal.

4. Two prisms having the same base are to each other as their altitudes.

5. Two regular pyramids are equal when the base and lateral edge of the one equal the base and lateral edge of the other.

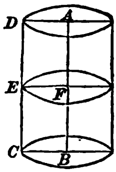
6. The surfaces of similar polyedrons are to each other as the squares of their homologous edges.

BOOK VII.

THE CYLINDER, THE CONE, AND THE SPHERE.

1. A CYLINDER is a volume which may be generated by the revolution of a rectangle about one of its sides as an axis.

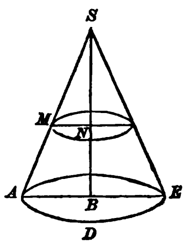
Thus, if the rectangle $ABCD$ be revolved around the side AB as an axis, it will generate the cylinder in the margin. The line AB is called the *axis;* the surface described by CD is called the *convex surface;* the circle BC is the *lower base;* the circle AD is the *upper base.*



It is evident that the circle described by the line EF perpendicular to the axis is equal to either base; hence, if a cylinder be cut by a plane parallel to the base, the section will be a circle equal to the base.

2. A CONE is a volume which may be generated by the revolution of a right-angled triangle about one of its sides adjacent to the right angle.

Thus, if the right-angled triangle SBA be revolved around SB as an axis, it will generate the cone $ADE—S$. The side SB is the *axis* of the cone; the circle described by AB is the base; the hypothenuse SA is the *slant height;* the surface generated by SA is the *convex surface.*

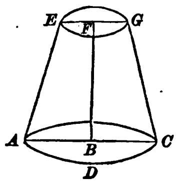


It is evident that the circle described by any line MN perpendicular to the

axis is a circle; hence, the section of a cone by a plane parallel to the base is a circle.

3. A FRUSTUM OF A CONE is the part which remains after cutting off the top with a plane parallel to the base.

Thus, *ADC—G* is the frustum of a cone; *FB* is its *altitude; EA* is its *slant height.* The frustum of a cone may be generated by the revolution of the trapezoid *ABFE.*

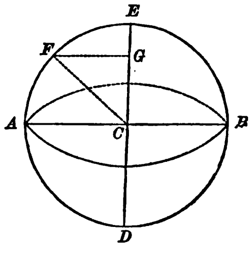


4. SIMILAR CYLINDERS or CONES are those whose axes are proportional to the radii or the diameters of their bases.

5. A prism may be *inscribed in a cylinder* by inscribing similar polygons with their sides parallel in each base, and uniting the vertices of the angles with straight lines. The cylinder is then said to circumscribe the prism.

6. A pyramid may be *inscribed in a cone;* and a frustum of a pyramid may be inscribed in the frustum of a cone.

7. A SPHERE is a volume bounded by a curved surface, every point of which is equally distant from a point within, called the *centre.*



The distance from the centre to the circumference is called the *radius.* The *diameter* is a line passing through the centre and limited at both extremities by the surface.

8. A SPHERICAL SECTOR is a volume generated by the revolution of a sector of a circle about the diameter. Thus, the revolution of *ACF* will generate a spherical sector.

9. A ZONE is a portion of the surface of a sphere in-

cluded between two parallel planes. The bounding lines of the zone are called its *bases;* the distance between the planes is its *altitude.*

10. A SPHERICAL SEGMENT is a portion of the sphere included between two parallel planes.

11. The CYLINDER, the CONE, and the SPHERE are the THREE ROUND BODIES of Geometry.

ANALYSIS.—This book treats of the *cylinder*, the *cone*, and the *sphere.* Its object is to find the *convex surface* and *volume* of each of these bodies, and also their *relation* to each other.

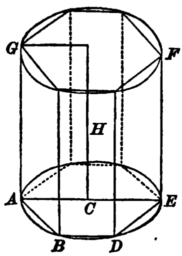
The method of treatment consists in regarding these volumes as polyedrons of an infinite number of sides. Thus, the cylinder is regarded as a right prism of an infinite number of sides, the cone as a right pyramid, and the sphere as a polyedron having its centre at the centre of the sphere.

CYLINDER, CONE, AND FRUSTUM.

THEOREM I.

The convex surface of a cylinder is equal to the circumference of its base multiplied by its altitude.

Let $ABDE$ be the base of a cylinder whose altitude is H; then will its convex surface be equal to circumference $CA \times H$.



For, inscribe in the cylinder a prism whose base is a regular polygon. Now, the convex surface of this prism will be equal to the perimeter of its base multiplied by its altitude (B. VI. Th. I.); and this is true whatever the number of sides; hence, it is true when the number of sides is infinite. But when the number of sides is infinite, the convex surface of the prism becomes the convex surface of the cylinder, the perimeter of the base of the prism becomes the circumference of the base of the cylinder, and the altitudes being the same, therefore, the convex surface of the cylinder equals the circumference of its base multiplied by its altitude.

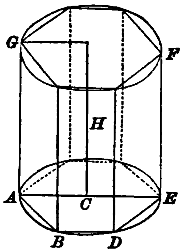
Cor. 1. Since the circumference of the base is $2\pi R$, the expression for the convex surface of a cylinder is $2\pi R \times H$.

Cor. 2. The convex surfaces of cylinders which have equal altitudes are to each other as the circumferences of their bases.

THEOREM II.

The volume of a cylinder is equal to the area of its base multiplied by the altitude.

Let *ABD—F* be a cylinder, whose altitude is *H;* then will its volume be equal to the area of its base multiplied by its altitude.



For, inscribe in the cylinder a prism whose base is a regular polygon. Now, the volume of this prism is equal to its base multiplied by its altitude (B. VI. Th. VI.), and this is true whatever the number of sides, and therefore true when the number of sides is infinite. But when the number of sides is infinite, the prism coincides with the cylinder in every respect; hence, the volume of the cylinder is equal to its base multiplied by its altitude. Therefore, etc.

Cor. 1. Since the area of the base is πR^2, the expression for the volume of a cylinder is $\pi R^2 \times H$.

Cor. 2. Cylinders are to each other as the products of their bases and altitudes. Cylinders having equal bases are to each other as their altitudes; cylinders having equal altitudes are to each other as their bases.

Cor. 3. Similar cylinders are to each other as the cubes of their altitudes, or of the radii of the bases. Let the pupil prove it.

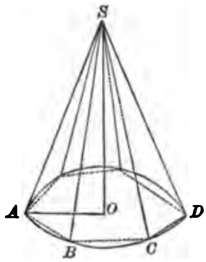
THEOREM III.

The convex surface of a cone is equal to the circumference of its base multiplied by one-half of the slant height.

Let *S—ABCD* be a cone whose base is *ABD* and slant height *SA;* then will its convex surface be equal to the

circumference of its base multiplied by one-half of its slant height.

For, inscribe in the cone a right pyramid. The convex surface of this pyramid is equal to the perimeter of its base multiplied by one-half of the slant height (B. VI. Th. VIII.); and this is true whatever the number of sides of the base; hence, it is true when the number of sides is infinite.



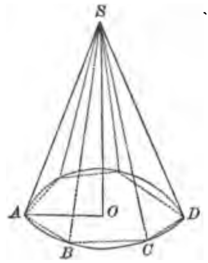
But when the number of sides is infinite, the pyramid coincides with the cone in every respect; hence, the convex surface of the cone is equal to the circumference of its base multiplied by one-half of the slant height.

Cor. 1. If S represents the slant height, the expression for the convex surface of a cylinder is $2\,\pi R \times \frac{1}{2}\,S$, or $\pi R \times S$.

THEOREM IV.

The volume of a cone is equal to the base multiplied by one-third of the altitude.

Let S—$ABCD$ be a cone whose base is $ABCD$ and altitude SO; then will its volume be equal to its base multiplied by one-third of its altitude.



For, inscribe in the cone a right pyramid. The volume of this pyramid is equal to the base $ABCD$ multiplied by one-third of its altitude SO (B. VI. Th. XIII.); and this is true whatever the number of sides of the base; hence, it is true when the number of sides is infinite. But when the number of

sides of the base is infinite, the pyramid becomes the cone; hence, the volume of a cone is equal to its base multiplied by one-third of its altitude. Therefore, etc.

Cor. 1. The expression for the volume of a cone is $\pi R^2 \times \frac{1}{3} H$, or, $\frac{1}{3} \pi R^2 \times H$.

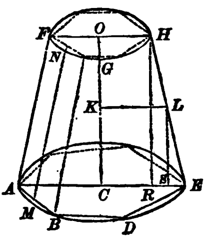
Cor. 2. A cone is one-third of a cylinder having an equal base and altitude.

Cor. 3. Cones are to each other as the products of their bases and altitudes; cones having equal bases are to each other as their altitudes; cones having equal altitudes are to each other as their bases.

THEOREM V.

The convex surface of a frustum of a cone is equal to one-half of the sum of the circumferences of the upper and lower bases multiplied by the slant height.

Let $ABDE$—H be a frustum of a cone, FA its slant height; then will its convex surface be equal to one-half of the sum of the circumferences of its two bases multiplied by its slant height.



For, inscribe within the frustum of a cone the frustum of a right pyramid. The convex surface of this frustum is equal to one-half the sum of the perimeters of its bases multiplied by the slant height (B. VI. Th. XV.); and this is true whatever the number of lateral faces; hence, it is true when the number of faces is infinite. But when the number of faces is infinite, the frustum of a pyramid becomes the frustum of a cone, the perimeters of its bases become the circumferences of the bases of the

frustum of the cone, and the slant height of the frustum of a pyramid becomes the slant height of the frustum of a cone; hence, the convex surface of the frustum of a cone equals one-half the sum of the circumferences of its bases multiplied by the slant height.

Cor. The expression for the convex surface of a frustum of a cone is $\frac{1}{2}(2\pi R + 2\pi R') \times S$, where R and R' represent the radii of the bases, and S the slant height.

Scholium. Through L, the middle point of HE, draw LK parallel to EC, and HR and LS perpendicular to EC; then, by similar triangles, since HL and LE are equal, RS and SE are equal; hence,

$$KL = \frac{1}{2}(CE + OH).$$

Multiplying this by 2π, we have,

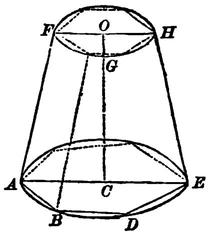
$$2\pi KL = \frac{1}{2}(2\pi CE + 2\pi OH);$$

that is, *circ.* KL equals $\frac{1}{2}$ of the sum of the circumferences of the two bases; hence, *the convex surface of the frustum of a cone, generated by the revolution of the line HE, is equal to the circumference of a circle generated by its middle point into the length of the line.*

THEOREM VI.

The volume of the frustum of a cone is equal to the sum of the volume of three cones, having for a common altitude the altitude of the frustum, and for bases the two bases of the frustum and a mean proportional between them.

Let $ABDE—H$ be a frustum of a cone, OC its altitude; then will its volume be equal to the sum of the volumes of three cones whose common altitude is OC, and whose bases are the two bases and a mean proportional between them.



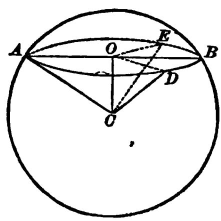
For, inscribe in the frustum the frustum of a right pyramid. The volume of this frustum is equal to the sum of the volumes of three pyramids having the common altitude of the frustum, and whose bases are the two bases of the frustum and a mean proportional between them (B. VI. Th. XVI.); and this is true whatever the number of lateral faces, and, hence, true when the number of faces is infinite. But when the number of lateral faces is infinite, the frustum of the pyramid becomes the frustum of a cone, and the three pyramids become cones; hence, the volume of the frustum of a cone equals the sum of the volumes of three cones, whose common altitude is the altitude of the frustum, and whose bases are the two bases of the frustum and a mean proportional between them.

Cor. The expression for the volume of a frustum of a cone is $(\pi R^2 + \pi r^2 + \pi R \times r) \times \frac{1}{3} H$.

THE SPHERE.

THEOREM VII.

Every section of a sphere made by a plane is a circle.



Let C be the centre of a sphere whose radius is CA, and ADB any section made by a plane; then will this section be a circle.

For, draw CO perpendicular to the section ADB, and draw the lines OD and OE to different points of the

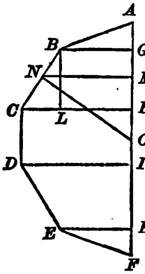
curve ADB; draw also the radii CD and CE. Then, since the radii CD and CE are equal, the lines OD and OE must be equal (B.I.Th.XIV.); hence, the section ADB is a circle. Therefore, etc.

Cor. If the plane pass through the centre of the sphere, the radius of the section will be equal to the radius of the sphere. The section is then called a *great circle.* All other sections are called *small circles.*

THEOREM VIII.

If a regular semi-polygon be revolved about a line passing through its centre and the vertices of two opposite angles, the surface generated by the semi-perimeter will be equal to the circumference of the inscribed circle multiplied by the axis.

Let $ABCDEF$ be a regular semi-polygon, AF the axis, ON the radius of the inscribed circle; then will the surface generated by the revolution of the semi-polygon be equal to *circ.* $ON \times AF$.



For, from the extremities of any side, as BC, draw BG and CH perpendicular to AF; from N, the middle point of BC, draw NM perpendicular to AF; draw also BL perpendicular to CH. Now, the surface described by BC is equal to *circ.* $MN \times BC$ (Th. V. S.). But, since the triangles BCL and NOM are similar, we have,

$$BC : BL \text{ or } GH :: ON : NM :: circ.\ ON : circ.\ NM;$$

hence, $$circ.\ NM \times BC = circ.\ ON \times GH;$$

that is, the surface generated by BC is equal to the circumference of the inscribed circle multiplied by the altitude GH; and the same may be shown for each of the other sides; hence, the surface described by the entire

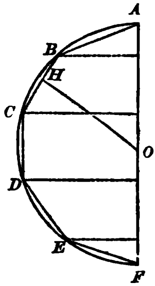
semi-perimeter is equal to the circumference of the inscribed circle multiplied by the sum of AG, GH, HI, etc., or the axis AF. Therefore, etc.

Cor. The surface described by any portion of the perimeter, as BCD, is equal to *circ.* $ON \times GI$.

THEOREM IX.

The surface of a sphere is equal to the circumference of a great circle multiplied by the diameter.

Let $ABCDEF$ be a semicircle, O its centre, and AF its diameter; then will the surface of the sphere generated by the revolution of the semi-circumference about the diameter be equal to *circ.* $OA \times AF$.



For, inscribe in the semi-circumference a regular semi-polygon. The surface described by the revolution of the polygon is equal to *circ.* $OH \times AF$ (Th. VIII.); and this is true whatever the number of sides; hence, it is true when the number of sides is infinite, in which case the volume becomes a sphere with the radius OA; hence, the surface of a sphere is equal to *circ.* $OA \times AF$. Therefore, etc.

Cor. 1. *The surface of a sphere is equal to four of its great circles.* For, *sur.* $=$ *circ.* $OA \times 2\,OA$; but *circ.* $OA = 2\pi OA$; hence, *sur.* $= 2\pi OA \times 2\,OA$; which gives *sur.* $= 4\pi OA^2$; but πOA^2 is the area of a great circle; hence, $4\pi OA^2$ is the area of *four* great circles.

Cor. 2. The expression for the surface of a sphere is $4\pi R^2$, or πD^2, in which R is the radius and D the diameter.

Cor. 3. The surfaces of spheres are to each other as the

squares of their radii or diameters. For, $sur.\ S = 4\,\pi R^2$, and $sur.\ s = 4\,\pi r^2$; hence,

$$S : s :: 4\,\pi R^2 : 4\,\pi r^2, \text{ or } R^2 : r^2.$$

Cor. 4. The surface of a zone is equal to the circumference of a great circle multiplied by its altitude.

Cor. 5. Zones on the same sphere, or on equal spheres, are to each other as their altitudes. A zone is to the surface of a sphere as the altitude of the zone is to the diameter of the sphere.

THEOREM X.

The volume of a sphere is equal to its surface multiplied by one-third of its radius.

For, conceive a regular polyedron to be inscribed in a sphere; this polyedron may be conceived as consisting of pyramids having their vertices at the centre of the sphere, and for bases the faces of the polyedron. Now, the volume of each of these pyramids is equal to its base multiplied by one-third of its altitude, and, their altitudes being equal, the volume of the polyedron will be equal to the sum of all their bases, which is the surface of the polyedron, multiplied by one-third of the common altitude; and since this is true whatever the number of faces of the polyedron, it is true when the number of faces is infinite, in which case the polyedron becomes the sphere, and the altitude of each pyramid becomes the radius of the sphere; hence, the volume of a sphere is equal to its surface multiplied by one-third of the radius.

Cor. 1. If we represent the volume of a sphere by *vol.* S, and the surface by *sur.* S, we will have,

$vol.\ S = sur.\ S \times \frac{1}{3} R$; and since $sur.\ S = 4\,\pi R^2$,

we have, $vol.\ S = 4\,\pi R^2 \times \frac{1}{3} R$; which, reduced,

gives, (1) $vol.\ S = \frac{4}{3}\pi R^3.$

But, $R = \frac{1}{2}D$, or $R^3 = \frac{1}{8}D^3$,

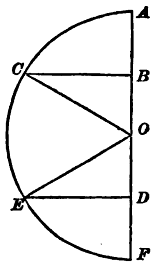
Hence, (2) $vol.\ S = \frac{1}{6}\pi D^3.$

Cor. 2. Spheres are to each other as the cubes of their radii, or diameters.

Cor. 3. The volume of a spherical sector or pyramid is equal to its base multiplied by one-third of the radius.

For the sector or pyramid may be conceived as consisting of an infinite number of pyramids having their vertices at the centre of the sphere, and the volume of the sum of these will be the sum of their bases multiplied by one-third of the radius.

Cor. 4. The volume of a spherical segment of one base and less than a hemisphere, as that generated by ACB revolving about AF, is equal to the volume of the spherical sector AOC minus the volume of the cone formed by OCB.



The volume of a spherical segment of one base and greater than a hemisphere, as AED, is equal to the volume of the spherical sector AOE plus the volume of the cone formed by EDO.

The volume of a spherical segment of two bases, as that generated by $BCED$, is equal to the volume of the sector, formed by COE, plus the volume of the cones formed by OCB and OED. If the points C and E fall on the same side of the centre, the last cone must be subtracted. The measure is as follows:

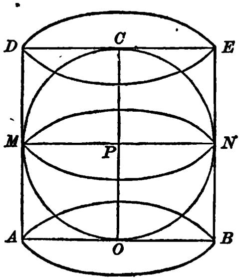
$Segment\ BCED = zone\ CE \times \frac{1}{3}OC + \pi\overline{BC}^2 \times \frac{1}{3}OB + \pi\overline{DE}^2 \times \frac{1}{3}OD.$

THEOREM XI.

The surface of a sphere is to the convex surface of the circumscribed cylinder, including its bases, as 2 to 3; and their volumes are to each other in the same ratio.

Let AE be a cylinder circumscribed about a sphere whose centre is P; then,

First. The surface of the sphere is to the entire surface of the cylinder as 2 is to 3.



For, the surface of the cylinder equals *circumference* $AO \times OC$ (Th. I.); that is, the circumference of a great circle of the sphere multiplied by the diameter of the sphere; but this is equal to the surface of a sphere (Th. IX.); hence, the surface of the cylinder equals the surface of the sphere; but the surface of the sphere equals four great circles; hence, the convex surface of the cylinder equals four great circles, and adding the two bases, we have the entire surface of the cylinder equal to six great circles; hence, the surface of the sphere is to the surface of the cylinder as 4 great circles is to 6 great circles, or as 4 to 6, or 2 to 3.

Second. The volume of the sphere is to the volume of the cylinder as 2 is to 3.

For, the volume of the sphere is $\frac{4}{3}\pi R^3$ (Th. X. C. I.), and the volume of the cylinder is $\pi R^2 \times CO$ (Th. II.), or $\pi R^2 \times 2R = \frac{6}{3}\pi R^3$; hence,

$$vol.\ S. : vol.\ cyl. :: \tfrac{4}{3}\pi R^3 : \tfrac{6}{3}\pi R^3, \text{ or,}$$
$$:: 4 : 6, \text{ or } 2 : 3.$$

Therefore, etc.

PRACTICAL EXAMPLES.

1. Required the convex surface and contents of a cylinder whose altitude is 16 inches, and diameter of the base 8 inches.

Ans. 402.12; 804.25.

2. Required the convex surface and volume of a cone whose altitude is 24 inches, and radius of the base 10 inches.

Ans. 816.816; 2513.28.

3. Required the convex surface of the frustum of a cone whose altitude is 36 inches, the radius of the upper base 6 inches, and lower base 21 inches. *Ans.* 3308.1048.

4. Required the volume of a frustum of a cone whose altitude is 9 feet, diameter of lower base 4 feet, and of upper base 2 feet.

Ans. 65.9736.

5. Required the surface and contents of a sphere whose diameter is 16 inches. *Ans.* 804.2496; 2144.6656.

6. The surface of a sphere is 1809.5616 square inches; required its diameter and its volume. *Ans.* D. = 24 inches.

7. The volume of a sphere is 113.0976 cubic inches; required its diameter and its surface. *Ans.* D. = 6 inches.

8. Given the volume of a sphere 268.0832 cubic inches; required the altitude of the circumscribing cylinder. *Ans.* 8 inches.

9. What is the surface of a zone of a single base whose altitude is 10 feet, the diameter of the sphere being 100 feet?

Ans. 3141.6 sq. ft.

10. Required the volume of a spherical segment of one base whose altitude is 2 feet, the diameter of the sphere being 8 feet.

Ans. 41.888 cubic feet.

11. Required the volume of a spherical segment whose greater diameter is 24 inches, less diameter 20 inches, and distance of bases 4 inches. *Ans.* 1566.6112 cubic inches.

THEOREMS FOR ORIGINAL THOUGHT.

1. Prove that two great circles of a sphere bisect each other.

2. Prove that every great circle divides the sphere into two equal parts.

3. Prove that the centres of a small circle and the sphere are in a line perpendicular to the small circle.

4. Prove that the radius of a small circle is less than the radius of the sphere.

5. Prove that circles whose planes are equidistant from the centre are equal.

6. Prove that the intersection of two spheres is a circle.

7. Prove that the arc of a great circle may be made to pass through any two points on the surface of a sphere.

8. Prove that if a cone and sphere be inscribed in a cylinder, that these bodies are to each other as 1, 2, and 3.

MISCELLANEOUS PROBLEMS.—PLANE FIGURES.

1. How many bricks 8 inches long and 4 inches wide will it take to pave a yard 20 feet by 16 feet? *Ans.* 1440.

2. How much will it cost to plaster a room whose length is 24 ft., width 18 ft., and height 12 ft., at 16 cts. a square yard? *Ans.* $25.60.

3. What is the difference in area between a rectangle 60 feet by 40 feet, and a square which has the same perimeter? *Ans.* 100 sq. ft.

4. What is the diagonal of a square whose area is equal to the area of a rectangle 16 inches by 25 inches? *Ans.* 28.28 inches.

5. The diagonal of a square is $\sqrt{50}$ inches; required the side of the square. *Ans.* 5 inches.

6. Required the diagonal of a room whose length is 48 feet, width 20 feet, and height 39 feet. *Ans.* 65 feet.

7. A vessel sailed north 20 miles, then west 80 miles, then north 60 miles, then west 70 miles; how far was it then from the point at which it started? *Ans.* 128.06 miles.

8. The gable ends of a house are 48 ft. wide, and the ridge-pole is 10 ft. above the eaves; required the length of the rafters. *Ans.* 26 ft.

9. Required the area of an isosceles triangle whose base is 20 feet, and each of its equal sides 15 feet. *Ans.* 111.803 square feet.

10. A flag-staff was broken, and fell, the broken part resting upon the upright, so that the end struck 48 feet from the foot; the upright part measured 36 feet; how long was the staff? *Ans.* 96 feet.

11. I wish to enclose a square rod in the form of an equilateral triangle; what must be the length of each side? *Ans.*

12. Given the area of a circle 19.635 square inches; required the diameter and circumference. *Ans.* D. = 5 inches.

13. The equal sides of an equilateral triangle are each 16 feet; what is the side of the inscribed square? *Ans.*

14. I have a plank 12 feet long which contains 15 square feet; what is the width of each end, if they are as 2 to 3? *Ans.* 12 in.; 18 in.

15. If the minute-hand of a clock is 6 inches long, over how much space does it pass in 40 minutes? *Ans.* 75.398 square inches.

16. What is the circumference of a circle whose diameter equals the diagonal of a square which contains 25 sq. rds.? *Ans.* 22.211112.

17. What is the diameter of a wheel which makes 200 revolutions in a minute, when the cars are going 30 miles an hour? *Ans.* $4\frac{1}{5}+$ feet.

18. A horse is fastened in a meadow, by a halter 20 feet long, to the top of a post 6 feet high; what is the area of the circle over which he can graze? *Ans.* 127.06 square yards.

19. Required the area of a circle in which the number expressing its area equals the number expressing its circumference.

Ans. 12.5664.

20. The area of a circular park is 4 acres; how long will it take to drive round it at the rate of 6 miles an hour? *Ans.* 2 min. 48 sec.

21. A circular garden containing 2 acres is bordered by a gravel walk of uniform width, which takes up $\frac{1}{4}$ of its area; required the width of the walk. *Ans.* 22.308 feet.

22. If the hour-hand of a clock is 4 inches long, and the minute-hand 6 inches, what is the difference of the surfaces over which they travel in an hour? *Ans.* 108.91 square inches.

MISCELLANEOUS PROBLEMS.—VOLUMES.

1. Required the surface of a brick 8 inches long, 4 inches wide, and 2 inches thick. *Ans.* 112 square inches.

2. Required the entire surface of a right pyramid whose base is a square 4 in. long, and the slant height 12 inches. *Ans.* 112 sq. in.

3. Required the entire surface of a cylinder whose altitude is 16 in., the radius of the base being 6 inches. *Ans.* 829.3824 sq. in.

4. Required the entire surface of a cone whose height is 16 feet, the radius of the base being 12 feet.

5. Required the surface and contents of a sphere inscribed in a cube whose edge is 20 inches; and also the space between them.

6. The surface of a sphere is 6.305 square feet; required its diameter and volume. *Ans.* Vol. 1.48868 cubic feet.

7. The volume of a sphere is 1.2411 cubic feet; required the diameter and surface. *Ans.* D. 16 inches.

8. The convex surface of a cylinder whose altitude is 14 feet is 116.666 square feet; required the diameter of its base. *Ans.* 2.65 ft.

9. What is the volume of a cylinder whose height is 20 feet, and the circumference of the base is 20 feet also? *Ans.* 636.64 feet.

10. The volume of a cylinder is 15.708 cubic feet; what is the altitude, if the diameter of the base is 2 feet? *Ans.* 5 feet.

11. The convex surface of a cone is 141.372 square feet, and the diameter of the base 4.5 feet; required the slant height and altitude.

Ans. 20 feet.

12. If a segment of 6 feet slant height be cut off of a cone whose slant height is 30 feet, the circumference of the base being 10 feet, what is the surface of the frustum? *Ans.* 144 square feet.

13. The convex surface of a frustum is 376.992 square feet, the slant height 20 feet, and the diameter of the less end 4 feet; what is the diameter of the greater end? *Ans.* 8 feet.

14. The volume of a cone is 8.83575 cubic feet, the altitude 15 feet; what is the diameter of the base? *Ans.* 18 inches.

15. The volume of a frustum of a cone is 65.9736 cubic feet, the diameter of one end is 4 feet and of the other 2 feet; required the altitude. *Ans.* 9 feet.

16. Required the entire surface of the frustum of a cone whose height is 12 feet, the radius of the lower base being 9 feet and the upper base 4 feet. *Ans.* 266 π.

17. Required the entire surface of the frustum of a pyramid whose bases are squares, the lower 9 feet, the upper 4 feet, on a side, the altitude being 12 feet. *Ans.* 415.68 sq. ft.

18. How far must a person ascend above the earth that he may see one-third of the surface? *Ans.* 2 times the radius.

MENSURATION.

MENSURATION OF LENGTHS AND SURFACES.

1. **Mensuration** is the science which treats of the measurement of geometrical magnitudes.

2. The **Area** of a figure is its quantity of surface; it is expressed by the number of times which it contains the unit of measure.

3. This *Unit of Measure* is a square whose side is some known length; as, an inch, a foot, etc.

4. The unit of surface has generally the same name as the linear unit; thus, if the linear unit is *one foot*, the surface unit is one *square foot*, etc.

5. Some superficial units have no corresponding linear unit of the same name; as, the *rood* and *acre*.

6. To refresh the memory, we give a few of the more important measures of surfaces.

1 rood = 40 perches, or square rods.
1 acre = 4 roods.
1 square mile = 640 acres.

Also,

1 chain = 100 links = 4 rods.
10 chains = 1 furlong.
1 square chain = 100×100 = 10,000 square links.
1 acre = 10 square chains = 100,000 square links.

THE TRIANGLE.

7. The AREA is found by the following rules:

RULE 1.—*Multiply the base by one-half of the altitude;* or,

RULE 2.—*Take half the sum of the sides, subtract from it each side separately, multiply the half sum and these remainders together, and take the square root of the product.*

1. What is the area of a triangular field whose base is 20 rods and altitude 16 rods? *Ans.* 2 acres.

2. Required the area of a triangle whose sides are 20, 30, and 40 chains respectively. *Ans.* 29 A. 8 P.

3. A man has a triangular garden whose sides are 150, 200, and 250 feet respectively; required the area. *Ans.* 1666.66 yards.

THE QUADRILATERAL.

8. PARALLELOGRAM.—The AREA is found as follows:

RULE.—*Multiply the base by the altitude.*

1. What is the area of a parallelogram 9 feet long and 7 feet wide? *Ans.* 63 square feet.

2. How many acres in a square field whose side is 70½ chains? *Ans.* 497 A. 4 P.

3. A man has a lot in the form of a rhombus, whose length is 333 feet and altitude 33.35 feet; required its area. *Ans.* 1233.95 square yards.

9. TRAPEZOID.—The AREA is found as follows:

RULE.—*Multiply one-half of the sum of the parallel sides by the altitude.*

1. Required the area of a trapezoid, one side being 192 inches and the other 96 inches, and altitude 12 feet. *Ans.* 144 square feet.

2. What is the area of a plank 24 feet long, 18 inches wide at one end and 12 inches at the other?

Ans. 30 square feet.

3. A farmer has a field in the form of a trapezoid, whose parallel sides are 95 and 75 rods respectively, and the perpendicular distance between them 65 rods; how much land in the field? *Ans.* 34 A. 2 R. 5 P.

10. TRAPEZIUM.—The AREA is found as follows:

RULE.—*Divide the trapezium into two triangles by a diagonal, find the area of each triangle, and take their sum.*

1. What is the area of a trapezium whose diagonal is 290 inches, and the altitudes of the triangles, the diagonal being the base, are 60 and 80 inches respectively?

Ans. 140 square feet, 140 square inches.

2. Required the area of a trapezium the lengths of whose sides are respectively 40, 60, 50, and 70 chains, and the diagonal 80 chains. *Ans.* 289 A. 1 R. 24 P.

POLYGONS OF ANY NUMBER OF SIDES.

11. REGULAR POLYGONS.—The AREA is found as follows:

RULE.—*Multiply half the perimeter by the perpendicular let fall from the centre on one of the sides.*

1. What is the area of a regular hexagon whose side is 14.6 feet and perpendicular 12.64 feet?

Ans. 61.5147 square yards.

2. Required the area of an octagon whose sides are 9.941 feet and its perpendicular 12 feet.

Ans. 477.168 square feet.

12. The following table shows the areas of ten regular polygons when the side is 1:—

Triangle	0.4330127	Octagon	4.8284271
Square	1.0000000	Nonagon	6.1818242
Pentagon	1.7204774	Decagon	7.6942088
Hexagon	2.5980762	Undecagon	9.3656404
Heptagon	3.6339124	Dodecagon	11.1961524

Now, since the areas of similar polygons are to each other as the squares of their homologous sides, to find the area of a regular polygon we have the following

RULE.—*Square the side of the polygon, and multiply by the tabular area set opposite the polygon.*

3. What is the area of a regular hexagon whose side is 5 inches long? *Ans.* 64.9519 square inches.

4. Required the area of an octagon whose sides are each 3 feet 4 inches. *Ans.* 53.649 square feet.

13. IRREGULAR POLYGON.—The AREA is found as follows:

RULE.—*Draw diagonals dividing the polygon into triangles, find the area of these triangles, and take the sum.*

1. In the irregular pentagon *ABCDE*, the diagonal *AC* is 24 inches, the diagonal *AD* is 18 inches, the altitude of the triangle *ABC* is 8 inches, of *ACD* is 10 inches, and of *AED* 6 inches; required the area.

Ans. 240 square feet.

2. In the irregular hexagon *ABCDEF*, the side *AB* is 268, *BC* 249, *CD* 310, *DE* 290, *EF* 199, and *AF* 246 links, and the diagonals *AC* 459, *CE* 524, and *AE* 326 links; required the area. *Ans.* 1 A. 2 R. 22 P. 13 yd. 47 ft.

THE CIRCLE.

14. The CIRCUMFERENCE is found by the following

RULE.—*Multiply the diameter by* 3.1416.

NOTE.—Hence, the diameter equals the circumference divided by 3.1416, or multiplied by .31831.

1. What is the circumference of a circle whose diameter is 50 inches? *Ans.* 157.08 inches.

2. A man has a circular fish-pond 32 rods in diameter; what is the distance around it? *Ans.* 100.5312 rods.

3. Required the diameter of a water-wheel whose circumference is 78.54 feet. *Ans.* 25 feet.

4. A man has a garden in the form of a circle, the diameter being 45 rods; what is the distance around it? *Ans.* 141.372 rods.

15. The LENGTH OF AN ARC, when its degrees and radius are given, is found as follows:

RULE.—*Multiply the number of degrees by the decimal* .01745, *and the product by the radius.*

1. The degrees in an arc are 45, and the radius 10; what is the length of the arc? *Ans.* 7.852.

2. What is the length of an arc of 32° 38′ 42″, the radius being 25 inches? *Ans.* 14.2441 inches.

16. When the *chord* and *chord of the half* arc are given.

RULE.—*From* 8 *times the chord of half the arc, subtract the chord of the whole arc, and divide the remainder by* 3.

1. The chord of an arc is 96 inches, and the chord of half the arc is 60 inches; what is the length of the arc? *Ans.* 128 inches.

2. The chord of an arc is 16 inches, and the diameter of the circle is 20 inches; what is the length of the arc? *Ans.* 18.5178 inches.

17. The AREA OF A CIRCLE is found as follows:

RULE I.—*Multiply the circumference by one-fourth of the diameter, or the square of the radius by* 3.1416.

RULE II.—*Multiply the square of the diameter by* .7854, *or the square of the circumference by* .07958.

(Let the pupil prove the last rule from the previous principles.)

1. What is the area of a circle whose diameter is 50 inches and circumference 157.08 inches?

Ans. $1963\frac{1}{2}$ square inches.

2. Required the area of a circle whose diameter is 18 inches. *Ans.* 254.4696 square inches.

3. What is the area of a circular garden whose circumference is 90 rods? *Ans.* 644.598 square rods.

18. The AREA OF A SECTOR is found as follows:

RULE.—I. *Multiply the arc by one-half the radius;* or,

II. *The sector is to the circle as the number of degrees in the sector is to* 360°.

1. What is the area of a circular sector whose arc contains 18°, the diameter of the circle being 6 feet?

Ans. 1.4137 square feet.

2. Required the area of a sector, the chord of half the arc being 30 inches, and the radius 50 inches.

Ans. 1523.45 square inches.

19. The AREA OF A SEGMENT is found as follows:

RULE.—*Find the area of the sector having the same arc, and also the area of the triangle formed by the chord of the segment and the radii of the sector.*

If the segment is greater than a semicircle, add the two areas; if less, subtract them.

1. Required the area of a segment whose height is 2 inches, and chord 20 inches. *Ans.* 26.878 square inches.

2. What is the area of a segment whose height is 18 inches, the diameter of the circle being 50 inches?

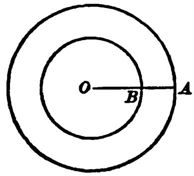
Ans. 632 sq. in.

3. Required the area of a segment whose arc is 180°, and radius of circle 12 feet. *Ans.* 226.1952.

20. The AREA OF A CIRCULAR RING is found as follows:

RULE.—*Find the difference of the squares of the radii, and multiply it by* 3.1416.

DEMONSTRATION.—Let the figure represent two circles having a common centre O; then the difference between them will be a circular ring. The area of circle OA is πOA^2, and of OB is πOB^2; the difference is $\pi OA^2 - \pi OB^2 = \pi (OA^2 - OB^2)$, which proves the rule.



1. What is the area of the circular ring when the diameters are 20 and 30?

Ans. 392.70.

2. A circular park 400 feet in diameter has a carriage-way around it 24 feet wide; required the area of the carriage-way. *Ans.* 3149.9776 square yards.

21. The SIDE OF AN INSCRIBED SQUARE is found thus:

RULE.—*Multiply the diameter by* .7071, *or multiply the circumference by* .2251.

1. What is the side of a square that can be cut out of a circular board whose diameter is 14 inches?

Ans. 9.899 inches.

2. How large a square can be cut out of a circular board whose circumference is 400 inches?

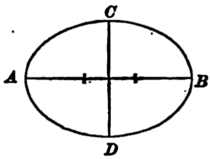
Ans. 90.04 inches.

THE ELLIPSE.

22. An ELLIPSE is a plane figure bounded by a curve, the sum of the distances from every point of which to two fixed points is equal to the line drawn through those points and terminated by the curve.

The two points are called *foci;* the line through the foci is the *transverse axis;* a line perpendicular to this through the centre is the *conjugate axis.*

23. The AREA is found by the following



RULE.—*Multiply half of the two axes together, and multiply that product by* 3.1416.

1. What is the area of an ellipse whose transverse axis is 20 inches and conjugate axis 16 inches?

Ans. 251.328.

2. Required the area of an elliptical mirror whose length is 6 feet and breadth 5 feet.

Ans. 23.562 square feet.

MENSURATION OF VOLUMES.

24. MENSURATION OF VOLUMES is the process of determining their surface and contents.

25. The CONTENTS of a volume is the number of times it contains a given *unit* of measure.

26. The UNIT OF MEASURE of a volume is a small cube whose dimensions are known.

MEASURES OF VOLUMES.

1 cubic foot	= 1728	cubic	inches.
1 " yard	= 27	"	feet.
1 " rod	= $4492\frac{1}{8}$	"	feet.
1 wine gallon	= 231	"	inches.
1 ale gallon	= 282	"	inches.
1 bushel	= 2150.42	"	inches.
1 cord	= 128	"	feet.

THE PRISM.

27. The CONVEX SURFACE OF A RIGHT PRISM is found thus:

RULE.—*Multiply the perimeter of the base by the altitude.*

To find the entire surface, we add the bases.

1. What is the convex surface of a triangular prism, the three sides of whose base are respectively 6, 7, and 8 inches, and the height 50 inches?

Ans. 1050 square inches.

2. What is the entire surface of a cube, the length of each side being 16 inches? *Ans.* $10\frac{2}{3}$ square feet.

3. What is the entire surface of the triangular prism given in the first problem? *Ans.* 1090.66 square inches.

28. The CONTENTS OF A PRISM are found thus:

RULE.—*Multiply the area of the base by the altitude of the prism.*

1. Required the contents of a cube whose sides are 30 inches. *Ans.* 15.625 cubic feet.

2. Required the contents of a square prism whose altitude is 27 feet, and the side of the base 4 feet?

Ans. 432 cubic feet.

3. Required the contents of a triangular prism whose altitude is 24 feet, the sides of the base being 3, 4, and 5 feet respectively. *Ans.* 144 cubic feet.

THE PYRAMID.

29. The CONVEX SURFACE OF A RIGHT PYRAMID is found thus:

RULE.—*Multiply the perimeter of the base by one-half the slant height.*

1. What is the convex surface of a triangular pyramid whose sides are 3, 4, and 5 feet, and slant height 20 feet?

Ans. 120 square feet.

2. Required the convex surface of a pentangular pyramid whose sides are each 5 feet, and slant height 60 feet.

Ans. 750 square feet.

30. The CONTENTS OF A PYRAMID are found thus:

RULE.—*Multiply the base by one-third of the altitude.*

1. Required the contents of a pyramid whose base is a hexagon, each side being 5 feet, and whose altitude is 20 feet. *Ans.* 433.013.

2. The pyramid of Cheops is 480 feet high, and the base is a square 763.4 feet on a side; required its solid contents.

Ans. 93244729$\frac{3}{5}$ cubic feet.

THE CYLINDER.

31. The CONVEX SURFACE and CONTENTS are found thus:

RULE 1.—*The surface equals the circumference of the base multiplied by the altitude.*

RULE 2.—*The contents equal the area of the base multiplied by the altitude.*

1. What is the convex surface of a cylinder 12 feet long and 6 feet in diameter? *Ans.* 226.1952 square feet.

2. Required the convex surface of a cylinder whose length is 20 feet and the diameter of the base 8 feet.

Ans. 502.656 square feet.

3. A man has a log 12 feet long and about 6$\frac{2}{3}$ feet in diameter; required its contents. *Ans.* 418.88 cubic feet.

4. The Winchester bushel is a cylinder containing 2150.42 cubic inches, its height being 8 inches; what is its diameter? *Ans.* 18$\frac{1}{2}$ inches.

THE CONE.

32. The CONVEX SURFACE and CONTENTS are found thus:

RULE 1.—*The surface equals the circumference of the base into one-half of the slant height.*

RULE 2.—*The contents equal the area of the base into one-third of the altitude.*

1. Find the convex surface and contents of a cone, the diameter of the base being 6 ft. and altitude 4 ft.

Ans. Sur. = 47.124.

2. Find the surface and contents of a cone whose slant height is 26 in. and radius of the base 10 in.

Ans. Vol. = 2513.28.

THE FRUSTUM OF A PYRAMID AND CONE.

33. The CONVEX SURFACE is found by the following

RULE.—*Find the sum of the perimeters or circumferences of the two bases, and multiply it by one-half of the slant height.*

1. Required the convex surface of the frustum of a square pyramid whose slant height is 24 ft., the side of the lower base 12 ft., and of the upper base 8 ft. *Ans.* 960 sq. ft.

2. Required the surface of a frustum of a cone whose slant height is 20 ft., the diameter of the lower base being 12 ft., and of the upper base 8 ft. *Ans.* 628.32 sq. ft.

34. The CONTENTS OF A FRUSTUM are found as follows:

RULE.—*Find the sum of the two bases and the square root of their product, and multiply this sum by one-third of the altitude of the frustum.*

NOTE.—In a frustum of a cone the following formula gives a shorter rule:—$V = \frac{\pi}{3}(R^2 + r^2 + R \cdot r) \times h$.

1. What is the amount of timber in a log which measures 40 feet in length, the radius of one base being 6 feet and of the other 3 feet? *Ans.* 2638.944 cubic feet.

2. Required the contents of the frustum of a regular hexagonal pyramid, the side of the greater end being 3 feet, that of the less 2 feet, the height being 24 feet.

Ans. 394.9075 cubic feet.

3. A cask, consisting of two equal conic frustums joined at their larger ends, has its bung diameter 30 inches, and its head diameter 20 inches; how many gallons of wine will it hold if $3\frac{1}{2}$ feet long? *Ans.* 90.44 gallons.

THE SPHERE.

35. The SURFACE OF A SPHERE is found as follows:

RULE.—*Multiply the diameter by the circumference;* or, *Square the radius, and multiply it by* 4 *and* 3.1416.

1. Required the surface of a sphere whose diameter is 17 inches. *Ans.* 6.305 square feet.

2. How many square miles on the surface of the earth, the diameter being about 7912 miles?

Ans. 196,663,355 square miles.

36. The SURFACE OF A ZONE is found as follows:

RULE.—*Multiply the height of a zone by the circumference of a great circle of the sphere.*

1. The diameter of a sphere is 25 feet, and the height of the zone 6 feet; what is the surface of the zone?

Ans. 471.24 square feet.

2. Required the surface of the torrid zone, the diameter of the earth being 7912 miles.

Ans. 78,293,218 square miles.

NOTE.—This is to be solved after the pupil has completed Trigonometry.

37. The CONTENTS OF A SPHERE are found as follows:

RULE.—*Multiply the surface by one-third of the radius;* or, *Multiply the cube of the diameter by* $\frac{1}{6}$ *of* 3.1416.

1. Required the contents of a sphere whose diameter is 17 inches. *Ans.* 2572.4468 cubic inches.

2. Required the contents of the planet Mars, the diameter being about 4500 miles. *Ans.* 47713050000.

38. The contents of a SPHERICAL SEGMENT OF ONE BASE are found thus:

RULE.—*Add the square of the height to three times the square of the radius of the base; multiply this sum by the height, and the product by* .5236; or, see Th. X. B. VII.

NOTE.—For the volume of a segment of two bases, see B. VII. Th. X. C. 4.

1. Required the contents of the segment of a sphere whose height is 4 inches, and radius of the base 8 inches.

Ans. 435.635 cubic inches.

2. Find the volume of either temperate zone, the diameter of the earth being 7912 miles.

Ans. 55,032,766,543 cubic miles.

NOTE.—The pupil will solve this after completing Trigonometry.

CYLINDRICAL RINGS.

39. A CYLINDRICAL RING is formed by bending a cylinder until the two ends meet. We find its surface by the following

RULE.—*To the thickness of the ring add the inner diameter; multiply this sum by the thickness of the ring, and the product by* 9.8696.

NOTE.—For contents, multiply the *sum* by the *square of ½ the thickness*, instead of the thickness, the other part of the rule being the same as for surface.

1. The thickness of a cylindrical ring is 4 inches, and the inner diameter 18 inches; what is the convex surface?

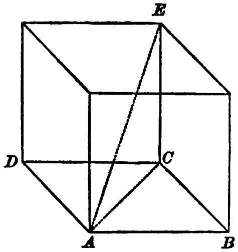
Ans. 868.52 square inches.

2. The thickness of a cylindrical ring is 2 inches, and the inner diameter 1 foot; required its contents.

Ans. 138.1744 cubic inches.

40. The SIDE OF AN INSCRIBED CUBE is found thus:

RULE.—*Multiply the diameter by* .57736, *or the radius by* 1.15472.



DEMONSTRATION.—Let the cube in the margin represent an inscribed cube; then will AE be the diameter of the sphere. Let the diameter be denoted by D, and the radius by R. Now, $\overline{AE}^2$ or $D^2 = AC^2 + \overline{CE}^2$; but $AC^2 = AB^2 + \overline{BC}^2$; hence, $D^2 = AB^2 + BC^2 + CE^2$, or, since the sides are equal, $D^2 = 3\ AB^2$, or, $D = AB \times \sqrt{3} = AB \times 1.73205$, and, consequently, $AB = D \times \frac{1}{1.73205} = D \times .57736$, or $AB = R \times 1.15472$.

1. Required the side of a cube that can be cut out of a sphere whose diameter is 16 inches. *Ans.* 9.23776.

2. Required the volume of a cube inscribed in a sphere whose circumference is 18.849552 inches.

Ans. 41.571219 cubic inches.

41. The VOLUME OF AN IRREGULAR BODY is found thus:

RULE.—*Immerse the body in a vessel of known dimensions, containing water; note the rise in the water, and calculate accordingly.*

1. A stone immersed in a cylindrical vessel 10 inches in diameter, raised the water 5 inches; required the volume of the stone. *Ans.* 392.70. cubic inches.

2. A man put a stone into a vessel 14 cubic feet in capacity, and it then required $2\frac{1}{2}$ quarts of water to fill the vessel; required the volume of the stone.

Ans. 13.9164 cubic feet.

MISCELLANEOUS PROBLEMS—PLANE FIGURES.

1. How many yards of paper that is 30 inches wide will it require to cover the wall of a room $15\frac{1}{2}$ feet long, $11\frac{1}{4}$ feet wide, and $7\frac{3}{4}$ feet high? *Ans.* 55.2833 yards.

2. A ladder 130 feet long, with its foot in the street, will reach on one side to a window 78 feet high, and on the other to a window 50 feet high; what was the width of the street? *Ans.* 224 feet.

3. The diameter of a circle is 4 feet; required the area of the inscribed equilateral triangle. *Ans.* $3\sqrt{3}$ square feet.

4. From a plank 16 inches broad, 6 square ft. are to be sawed off; at what distance from the end must the line be struck? *Ans.* $4\frac{1}{2}$ feet.

5. The ball on the top of a church is 6 feet in diameter; what did the gilding of it cost, at 8 cents per square inch? *Ans.* $1302.884.

6. The area of an equilateral triangle, whose base falls on the diameter and its vertex in the middle of the arc of a semicircle, is 100 square feet; what is the diameter of the semicircle? *Ans.* 26.32148.

7. The cost of paving a semicircular plot of ground, at 20 cents a square foot, amounted to $20; required its diameter. *Ans.* 11.2864.

8. A gentleman has a garden 80 feet long and 60 feet wide; what must be the width of a walk extending around the garden, which shall occupy one-half of the ground? *Ans.* 10 feet.

9. Required the perimeter of a regular dodecagon which shall contain the same area as a circle whose circumference is 1000 feet. *Ans.* 1011.67 feet.

10. If a horse tied to a post in the centre of a field by a rope 1 chain 78 links can graze upon an acre, what length of rope would allow it to graze upon $5\frac{4}{9}$ acres? *Ans.* 4 chains $15\frac{1}{3}$ links.

11. A has a circular garden which is 20 rods, and B has a circular garden whose area is $6\frac{1}{4}$ times as great; what is the diameter of B's garden? *Ans.* 50 rods.

12. A has a circular garden, and B a square one; the distance around each is 64 rods; which contains the most land, and how much? *Ans.* 69.948 square rods.

13. Atherton has a circular garden and Fell has a square one, and

they contain 4 acres; how much farther around is one than the other? *Ans.* 11.512 rods.

14. Mr. Thompson has a square yard containing $\frac{1}{10}$ of an acre; he makes a gravel walk around it which occupies $\frac{15}{64}$ of the whole yard; what is the width of the walk? *Ans.* 4 feet $1\frac{1}{2}$ inches.

15. A general, attempting to draw up his division in the form of a square, found he lacked 100 men to complete the square; he then received a reinforcement of five companies of 100 men each, and found he could increase the side of the square by 3 men and have 1 man remaining; how many men had he at first? *Ans.* 4125 men.

VOLUMES.

1. The volume of a sphere is 606.132 cubic feet; required its diameter. *Ans.* 10.5 feet.

2. The edge of a cube is 6 feet; what is the volume of a sphere that may be inscribed within it? *Ans.* 113.0976 cubic feet.

3. I have a cistern in the form of the frustum of a cone, its top diameter being 12 feet, its bottom diameter 9 feet, and its depth 5 feet; how many barrels of water will it contain? *Ans.* 56.24 barrels.

4. Bunker Hill Monument is 220 feet high, 30 feet square at the base, and 15 feet at the vertex; what is its volume? *Ans.* 115500 cubic ft.

5. Mr. Wilson has a pond which covers 100 acres, the average depth being 10 feet; how many cubic feet of water does it contain? *Ans.* 43560000 cubic feet.

6. A man has a log of wood 20 ft. long, the larger end being 3 ft. in diameter, and the smaller 2 ft.; required the contents of the largest square stick, 20 ft. long, that can be sawed out of it. *Ans.* $63\frac{1}{3}$ cubic feet.

7. A bushel measure is $18\frac{1}{2}$ inches in diameter and 8 inches deep; what should be the dimensions of a measure of similar form to contain 64 bushels? *Ans.* Diameter, 74 inches; depth, 32 inches.

8. Mr. Benson can dig a shaft 5 feet each way in one day; how long will it take him to dig a shaft 20 feet each way? *Ans.* 64 days.

9. A man has a square garden 100 feet long, and wishes to make a gravel walk half-way around it; what will be the width of the walk if it takes up one-half of the garden? *Ans.* 29.289 feet.

10. A wishes to enclose his garden, which is 100 feet long and 80 feet wide, with a ditch 4 feet wide; how deep must it be dug that the soil taken out may raise the surface 1 foot? *Ans.* 5.319 feet.

11. A cubic foot of brass is to be drawn into a wire $\frac{1}{30}$ of an inch in diameter; required the length of the wire, supposing there is no loss of metal in the process. *Ans.* 31.252 miles.

12. Mr. Bonnycastle mentions a globe whose volume and surface are represented by the same number; what was the diameter of this globe? *Ans.* 6.

13. Mr. Haswell requires the weight of an iron shell 4 inches in diameter, the thickness of the metal being 1 inch, estimating a cubic inch of iron at $\frac{1}{4}$ of a pound. *Ans.* 7.3304 pounds.

14. Bunker Hill Monument is 220 feet high, the lower base being 30 feet square, the upper 15 feet square; through its centre runs a cylindrical opening 15 feet in diameter at the bottom and 11 feet at the top; how many cubic feet of material in the monument?

Ans. 86068.444 cubic feet.

15. A gentleman has a bowling-green 300 feet long and 200 feet broad, which he wishes to raise 1 foot higher by means of the earth that is to be taken from a ditch that is to go around it; to what depth must the ditch be dug, supposing its breadth to be 8 feet?

Ans. 7 feet 3.21 inches.

16. A man having a garden 100 feet long and 80 feet broad, wishes to make a gravel walk half-way around it; what will be the width of the walk if it takes up one-half of the garden? *Ans.* 25.9688 feet.

17. Three persons having bought a sugar loaf, want to divide it equally among them by sections parallel to the base; what is the altitude of each person's share, supposing the loaf is a cone 20 inches high?

Ans. 13.867 upper part; 3.604 middle; 2.528 lower.

Suggestion.—Solve it by the principle of similar cones being to each other as the cubes of their altitudes.

Note.—Several of these problems are from the old writers on Mensuration. For more methods and exercises, see Bonnycastle's and Haswell's works on Mensuration.

Began Nov., 9th, 1870.

ELEMENTS OF TRIGONOMETRY.

INTRODUCTION.

LOGARITHMS.

1. Logarithms are a species of numbers used to abbreviate Multiplication, Division, Involution, and Evolution.

2. The *logarithm of a number* is the exponent denoting the power to which a fixed number must be raised in order to produce the first number.

3. This *fixed number* is called the *base* of the system. The base of the common system is 10.

4. Raising 10 to different powers, we have,

$10^0 = 1$	; hence,	0	is the log of	1;
$10^1 = 10$	"	1	"	10;
$10^2 = 100$	"	2	"	100;
$10^3 = 1000$	"	3	"	1000;
etc.				

5. From this we have the following principles:

Prin. 1. *The logarithm of a number between* 1 *and* 10 *is between* 0 *and* 1, *and is, therefore, a decimal.*

Prin. 2. *The logarithm of a number between* 10 *and* 100 *is between* 1 *and* 2, *and is, therefore,* 1 *and a decimal.* Thus, it has been found that the log. of 76 is 1.880814.

Prin. 3. *The logarithm of a number between* 100 *and* 1000 *is between* 2 *and* 3, *and is, therefore,* 2 *and a decimal.* Thus, the log. of 458 is 2.660865.

6. When the logarithm consists of an integer and a decimal,

the integer is called the *characteristic*, and the decimal part the *mantissa*. Thus, in 2.660865 the 2 is the characteristic, and .660865 is the mantissa.

PROPERTIES OF LOGARITHMS.

PRIN. 1.—*The characteristic is always one less than the number of integral places in the number.*

For, from Art. 4, we see that the log. of 100 is 2, the log. of 1000 is 3, and of any number between 100 and 1000 it is 2 and a decimal; hence, the characteristic is one less than the number of integral places.

PRIN. 2.—*The logarithm of the base is* 1, *and the logarithm of* 1 *is zero.*

For, since $10^1 = 10$, the log. of 10 is 1; and since $10^0 = 1$, the logarithm of 1 is 0.

PRIN. 3.—*The characteristic of the logarithm of a decimal is negative, and is numerically one greater than the number of ciphers between the decimal point and the first significant figure.*

For, if we raise the base, 10, to powers which give decimals, we will have,

$$\begin{array}{lll} 10^0 = 1 & ;\text{ hence, } \log 1 & = 0; \\ 10^{-1} = .1 & \quad\text{"}\quad \log .1 & = -1; \\ 10^{-2} = .01 & \quad\text{"}\quad \log .01 & = -2; \\ 10^{-3} = .001 & \quad\text{"}\quad \log .001 & = -3; \\ \quad\text{etc.} & & \quad\text{etc.} \end{array}$$

which proves the principle. Thus, the log. of .458 is $\bar{1}.660865$.

PRIN. 4.—*The logarithm of the product of two numbers is equal to the sum of the logarithms of those numbers.*

For, let M and N be any two numbers, and m and n their logarithms; then we shall have, according to the definition,

$$10^m = M, \qquad 10^n = N.$$

Multiplying these equations, member by member, we have,

$$10^{m+n} = M \times N.$$

Hence, $\log (M \times N) = m + n;$ or, $= \log M + \log N.$

PRIN. 5.—*The logarithm of the quotient of two numbers equals the difference of the logarithms of those numbers.*

For, from the definition, we have,

$$10^m = M, \qquad 10^n = N.$$

Dividing the first by the second, we have,

$$10^{m-n} = \frac{M}{N}.$$

Hence, $\log\left(\frac{M}{N}\right) = m - n$, or, $= \log M - \log N$.

PRIN. 6.—*The logarithm of any power of a number is equal to the logarithm of the number multiplied by the exponent of the power.*

For, since

$$10^m = M,$$

if we raise both members to the nth power, we have,

$$10^{mn} = M^n.$$

Hence, $\log M^n = mn$, or, $= \log M \times n$.

PRIN. 7.—*The logarithm of the root of any number is equal to the logarithm of the number divided by the index of the root.*

For, since

$$10^m = M,$$

if we take the nth root of both members, we have,

$$10^{\frac{m}{n}} = \sqrt[n]{M}.$$

Hence, $\log \sqrt{M} = \frac{m}{n}$, or, $\log M \div n$.

PRIN. 8.—*The logarithm of the product of any number multiplied by* 10 *is equal to the logarithm of the number increased by* 1.

Suppose $\log M = m$; then, by Prin. 4,

$$\log (M \times 10) = \log M + \log 10. \text{ But } \log 10 = 1;$$

Hence, $\log (M \times 10) = m + 1$.

Thus, $\log (76 \times 10) = 1.880814 + 1$; or, $\log 760 = 2.880814$.

PRIN. 9.—*The logarithm of the quotient of any number divided by* 10 *is equal to the logarithm of the number diminished by* 1.

Suppose $\log M = m$; then, by Prin. 5,

$$\log (M \div 10) = \log M - \log 10; \text{ from which}$$
$$\log (M \div 10) = m - 1.$$

Thus, $\log (458 \div 10) = 2.660865 - 1;$

or, $\log 45.8 = 1.660865.$

7. The following examples will illustrate Principles 1, 3, 8, and 9.

$$\begin{array}{lll} \log 234 & \text{is} & 2.369216, \\ \log 23.4 & \text{“} & 1.369216, \\ \log 2.34 & \text{“} & 0.369216, \\ \log .234 & \text{“} & \bar{1}.369216, \\ \log .0234 & \text{“} & \bar{2}.369216. \end{array}$$

From this, we see that when we change the place of the decimal point we change the characteristic, but do not change the decimal part of the logarithm.

The minus sign is written over the characteristic, showing that it only is negative.

TABLE OF LOGARITHMS.

8. A TABLE OF LOGARITHMS is a table by means of which we can find the logarithms of numbers, or the numbers corresponding to given logarithms.

9. In the annexed table the entire logarithms of the numbers up to 100 are given. For numbers greater than 100 the mantissa alone is given; the characteristic being found by Prin. 1.

10. The numbers are placed in the column on the left, headed N; their logarithms are opposite, on the same line. The first two figures of the mantissa are found in the first column of logarithms.

11. The column headed D shows the average differences of the ten logarithms in the same horizontal line. This difference is found by subtracting the logarithm in column 4 from that in column 5, and is very nearly the mean or average difference.

TO FIND THE LOGARITHM OF ANY NUMBER.

12. *To find the logarithm of a number of* ONE *or* TWO *figures.*

Look on the first page of the table, in the column headed N, and opposite the given number will be found its logarithm. Thus,

the logarithm of 25 is 1.397940,
" " 87 is 1.939519.

13. *To find the logarithm of a number of* THREE *figures.*

Look in the table for the given number; opposite this, in column headed 0, will be found the decimal part of the logarithm, to which we prefix the characteristic 2, Prin. 1. Thus,

the logarithm of 325 is 2.511883,
" " 876 is 2.942504.

14. *To find the logarithm of a number of* FOUR *figures.*

Find the three left-hand figures in the column headed N, and opposite to these, in the column headed by the fourth figure, will be found four figures of the logarithm, to which two figures from the column headed 0 are to be prefixed. The characteristic is 3, Prin. 1. Thus,

the logarithm of 3456 is 3.538574,
" " 7438 is 3.871456.

15. In some of the columns, *small dots* are found in the place of figures: these dots mean zeros, and should be written zeros. If the four figures of the logarithm fall where zeros occur, or if, in passing back from the four figures found to the zero column, any of these *dots are passed over*, the two figures to be prefixed must be taken from the line just below. Thus,

the logarithm of 1738 is 3.240050,
" " 2638 is 3.421275.

16. *To find the logarithm of a number of* MORE THAN FOUR *figures.*

Place a decimal point after the fourth figure from the left hand, thus changing the number into an integer and a decimal. Find the mantissa of the entire part by the method just given. Then

from the column headed D take the corresponding *tabular difference*, multiply it by the decimal part, and add the product to the mantissa already found; the result will be the mantissa of the given number. The characteristic is determined by Prin. 1.

If the decimal part of the product exceeds .5, we add 1 to the entire part; if less than .5, it is omitted.

EXAMPLES.

1. Find the logarithm of 234567.

SOLUTION.—The characteristic is 5, Prin. 1. Placing a decimal point after the fourth figure from the left, we have 2345.67. The decimal part of the logarithm of 2345 is .370143; the number in column D is 185; and $185 \times .67 = 123.95$, and since .95 exceeds .5, we have 124, which, added to .370143, gives .370267; hence, log $234567 = 5.370267$.

2. Find the logarithm of 4567. *Ans.* 3.659631.
3. Find the logarithm of 3586. *Ans.* 3.554610.
4. Find the logarithm of 11806. *Ans.* 4.072102.
5. Find the logarithm of .4729. *Ans.* $\bar{1}.674769$.
6. Find the logarithm of 29.337. *Ans.* 1.467416.

17. *To find the number corresponding to a given logarithm.*

1. Find the *two left-hand* figures of the *mantissa* in the column headed 0, and the other four, if possible, in the same or some other column, on the same line; then, in column N, opposite to these latter figures, will be found the *three left-hand* figures, and at the top of the page the other figure of the required number.

2. *When the exact mantissa is not given in the table*, take out the four figures corresponding to the *next less mantissa* in the table; subtract this mantissa from the given one; divide the remainder, with ciphers annexed, by the number in column D, and annex the quotient to the four figures already found.

3. Make the number thus obtained correspond with the characteristic of the given logarithm, by pointing off decimals or annexing ciphers.

EXAMPLES.

1. Find the number whose logarithm is 5.370267.

SOLUTION.—The mantissa of the given logarithm is . . .370267
The mantissa of the next less logarithm of the table is . .370143
and its corresponding number is 2345.
Their difference is 124
The tabular difference is 185
The quotient is . . . 185)124.00(.67
Hence, the required number is . 234567.

NOTE.—If the characteristic had been 2, the number would have been 234.567; if it had been 7, the number would have been 23456700; if it had been $\bar{2}$, the number would have been .0234567, etc.

2. Find the number whose logarithm is 3.659631.

Ans. 4567.

3. Find the number whose logarithm is 2.554610.

Ans. 358.6.

4. Find the number whose logarithm is 1.072102.

Ans. 11.806.

5. Find the number whose logarithm is $\bar{2}.674769$.

Ans. .04729.

6. Find the number whose logarithm is $\bar{3}.065463$.

Ans. .0011627.

MULTIPLICATION BY LOGARITHMS.

18. From Prin. 4, for the multiplication of numbers by means of logarithms, we have the following

RULE.—*Find the logarithms of the factors, take their sum, and find the number corresponding to the result; this number will be the required product.*

NOTE.—The term *sum* is used in its algebraic sense. Hence, when any of the characteristics are negative,—the mantissa is always positive,—we take the difference between the sums of the positive and negative characteristics, and prefix to it the sign of the greater. If any thing is to be carried from the addition of the mantissas, it must be added to a positive characteristic, or subtracted from a negative one.

EXAMPLES.

1. Multiply 35.16 by 8.15.

SOLUTION. log 35.16 = 1.546049
log 8.15 = 0.911158

2.457207
457125

Product, 286.554

152)82.00(.54

2. Find the product of .7856, 31.42. *Ans.* 24.6835.

3. Find the product of 31.42, 56.13, and 516.78. *Ans.* 911393.7.

4. Find the product of 31.462, .05673, and .006785. *Ans.* .01211168.

5. Product of .06517, 2.16725, .000317, and 42.1234. *Ans.* .001886.

6. Product of 2.3456, .00314, 123.789, .00078, and 67.105. *Ans.* .04772076.

DIVISION BY LOGARITHMS.

19. From Prin. 5, to divide by means of logarithms, we have the following

RULE.—*Find the logarithms of the dividend and divisor, subtract the latter from the former, and find the number corresponding to the result; this number will be the required quotient.*

NOTE.—The term *subtract* is here used in its algebraic sense; hence, we must subtract according to the principles of algebra.

EXAMPLES.

1. Divide 783.5 by 6.25.

SOLUTION. log 783.5 = 2.894039
log 6.25 = 0.795880

2.098159
.097951

Quotient, 125.36

346)208(6

2. Divide 272.636 by 6.37. *Ans.* 42.8.

3. Divide 50.38218 by 67.8. *Ans.* .7431.

4. Divide 155 by .0625. *Ans.* 2480.

ARITHMETICAL COMPLEMENT.

20. The operation of division when combined with multiplication is somewhat simplified by using the principle of the *arithmetical complement.*

21. The ARITHMETICAL COMPLEMENT of a logarithm is the result arising from subtracting the logarithm from 10. Thus, the arithmetical complement of the logarithm 5.623427 is 10 — 5.623427, or 4.376573.

22. The arithmetical complement may be written directly from the table, *by subtracting each figure of the logarithm from* 9, *except the right-hand figure, which must be taken from* 10. This is the same as subtracting the logarithm from 10.

23. We will now prove that *the difference between two logarithms is equal to the first logarithm, plus the arithmetical complement of the second, minus* 10.

Let $a =$ the first logarithm,
$b =$ the second logarithm,
and $c = 10 - b =$ arith. comp. of b.
The difference is $a - b$.
But, $-b = c - 10$.
Hence, $a - b = a + c - 10$,
which proves the principle.

24. Hence, to divide by means of the arithmetical complement, we have the following

RULE.—*Add the arithmetical complement of the logarithm of the divisor to the logarithm of the dividend, subtract* 10, *and find the number corresponding to the difference, this number will be the required quotient.*

EXAMPLES.

1. Divide 256.3 by 45.32.

SOLUTION.	log 856.3 . .	2.932626
	(a. c.) log 45.32 . .	8.343710
Quotient,	18.8945	1.276336

2. Divide 0.3156 by 78.35.

log 0.3156 . . .	$\bar{1}.499137$
(a. c.) log 78.35 . . .	8.105961
Quotient, .004028	$\bar{3}.605098$

3. Divide 3.7521 by 18.346. *Ans.* .204519.

4. Divide 483.72 by .30751. *Ans.* 1573.02.

5. Multiply 32.16 by 7.856, and divide the product by 45.327. *Ans.* 5.574.

6. Divide the product of 31.57 and 123.4 by the product of 316.2 and .0316. *Ans.* 389.8884.

7. Find by logarithms the first term of the proportion, $x : 73.15 :: 48.16 : 3167$. *Ans.* 1.11237.

INVOLUTION BY LOGARITHMS.

25. From Prin. 6, to raise a number to any power, we have the following

RULE.—*Find the logarithm of the number, multiply it by the exponent of the power, and find the number corresponding to the result.*

EXAMPLES.

1. Find the 4th power of 45.

SOLUTION.

$$\log 45 = 1.653213$$
$$\underline{\qquad\qquad 4}$$
Power, 4100625 6.612852

2. Find the cube of 0.65. *Ans.* 0.2746.

3. Find the 6th power of 1.037. *Ans.* 1.243.

4. Find the 7th power of .4797. *Ans.* 0.00584~~2~~5

EVOLUTION BY LOGARITHMS.

26. From Prin. 7, to extract any root of a number, we have the following

RULE.—I. *Find the logarithm of the number, divide it by the index of the root, and find the number corresponding to the result.*

II. *If the characteristic is negative and not divisible by the index of the root, add to it the smallest negative number that will make it divisible, prefixing the same number with a plus sign to the mantissa.*

EXAMPLES.

1. Find the square root of 576.

SOLUTION. $\log 576 = 2.760422$

$$2.760422 \div 2 = 1.380211$$

Hence, the root is 24.

2. Find the fourth root of .325.

SOLUTION. $\log .325 = \bar{1}.511883 = \bar{4} + 3.511883.$

Then $(\bar{4} + 3.511883) \div 4 = \bar{1}.877971$

Hence, the quotient is, .75504.

3. Find the fifth root of .0625. *Ans.* .574348.

4. Find the cube root of 7. *Ans.* 1.9129.

5. Find the fifth root of 5. *Ans.* 1.3797.

6. Find the tenth root of 8764.5. *Ans.* 2.479.

CALCULATION OF LOGARITHMS.

The pupil will by this time naturally inquire how these logarithms are calculated. This we have not room to explain here; in fact, an explanation of the modern methods would be almost too difficult for the majority of pupils who study this book. Only a general idea can here be given.

In computing logarithms, it is only necessary to calculate the logarithms of prime numbers, since the logarithms of composite numbers may be obtained by adding the logarithms of their prime factors.

The logarithms of the prime numbers were first computed by com-

paring the geometrical and arithmetical series, 1, 10, 100, etc., and 0, 1, 2, etc., and finding geometrical and arithmetical means; the arithmetical mean being the logarithm of the corresponding geometrical mean. This method was exceedingly laborious, involving so many multiplications and extractions of roots.

The method now generally used is that of series, by which the computations are much more easily made. The following formula is derived by algebraic reasoning.

$$\log (1 + x) = A \left(\frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \text{etc.}\right)$$

In this the quantity A is called the *modulus*, which in the Napierian system is *unity*. The series, when A is *one*, put in a more convenient form, becomes,

$$\log. (z + 1) - \log. z = 2\left(\frac{1}{2z + 1} + \frac{1}{3(2z+1)^3} + \frac{1}{5(2z+1)^5} + \text{etc.}\right)$$

From which, knowing the logarithm of any number, we readily find the logarithm of the next larger number. The pupil will be interested in finding logarithms by this formula. Begin with 2, in which $z = 1$.

The logarithm found will be the Napierian logarithm, and this multiplied by 0.434294 will give the common logarithm.

PLANE TRIGONOMETRY.

DEFINITIONS AND PRIMARY PRINCIPLES.

1. Plane Trigonometry is the science which treats of the solution of plane triangles.

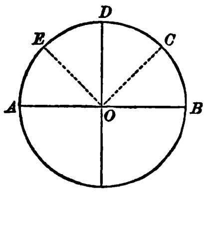
2. The Solution of a triangle is the operation of finding the unknown parts when a sufficient number of the known parts are given.

3. In every triangle there are six parts; *three sides* and *three angles.* These parts are so related that when three of the parts are given, one being a side, the other parts may be found.

4. An angle is measured, as we have previously seen, by the arc included between its sides, the centre of the circumference being at the vertex of the angle.

5. For measuring angles, as has already been explained, the circumference is divided into 360 equal parts, called degrees; each degree into 60 equal parts, called minutes, etc.

6. A Quadrant is one-fourth of the circumference of a circle; hence, if two lines be drawn through the centre of a circle at right angles to each other, they will divide the circumference into four quadrants. Each quadrant contains 90°.



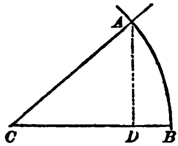
7. The Complement of an arc is 90° minus the arc; thus, DC is the complement of BC; also, the angle DOC is the complement of BOC.

8. The Supplement of an arc is 180° minus the arc; thus,

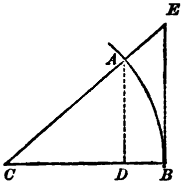
AE is the supplement of the arc *BDE;* also, the angle *AOE* is the supplement of the angle *BOE.*

9. In Trigonometry, instead of comparing the angles of triangles, or the arcs which measure them, we compare certain lines, called *functions* of the arcs. A function of a quantity is something depending upon the quantity for its value. These functions are the *sine*, *cosine*, *tangent*, *cotangent*, *secant*, and *cosecant.*

10. Thus, instead of reasoning with the angle *ACB*, or the arc *AB*, which measures it, we draw the perpendicular *AD*, and use the lines *AD* and *CD*. The line *AD* is called the *sine* of the arc or angle; the line *CD* is called the *cosine* of the arc or angle.

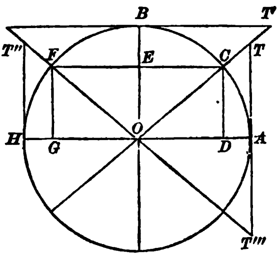


11. If we draw *BE* perpendicular to *CB*, meeting *CA* produced in *E*, the line *BE* is called the *tangent* of the angle, and the line *CE* is called the *secant.*



12. In comparing the sides and angles, these lines, we say, are used instead of the angles or the arcs. The necessity for such lines is evident, since we could not compare the sides, which are *straight lines*, with the *angles*, or the *curve lines*, which measure them.

We will now represent these lines in the first and second quadrants.



13. The SINE of an arc is the perpendicular let fall from one extremity of the arc on the diameter which passes through the other extremity. Thus, *CD* is the sine of the arc *AC.*

14. The COSINE of an arc is the sine of its complement; or it is the distance between the foot of the sine and the centre of the circle; thus, *CE* or *OD* is the cosine of the arc *AC.*

15. The TANGENT of an arc is a line which is perpendicular to the radius at one extremity of the arc, and limited by a line passing through the centre of the circle and the other extremity; thus, AT is the tangent of AC.

16. The COTANGENT of an arc is equal to the tangent of the complement of the arc; thus, BT' is the cotangent of AC.

17. The SECANT of an arc is a line drawn from the centre of the circle through one extremity of the arc, and limited by a tangent at the other extremity; thus, OT is the secant of AC.

18. The COSECANT of an arc is the secant of the complement of the arc; thus, OT' is the cosecant of AC.

19. The sine, cosine, tangent, cotangent, etc. of an arc are indicated as follows:

$\sin AC;$	$\tan AC;$	$\sec AC;$
$\cos AC;$	$\cot AC;$	$\mathrm{cosec}\, AC.$

20. PRINCIPLES.—From the definitions now given, we can readily derive the following simple principles.

1. *The sine of an arc equals the sine of its supplement, and also the cosine of an arc equals the cosine of its supplement.*

DEM.—Take the arc ABF; its sine is FG, its supplement is FH, and the sine of its supplement is FG. Hence, its sine equals the sine of its supplement. Its cosine is GO, which is also the cosine of FH. Hence, etc.

2. *The tangent and cotangent of an arc are respectively equal to the tangent and cotangent of the supplement of the arc.*

DEM.—The tangent of the arc ABF is AT''', and the tangent of its supplement FH is HT'', and, by similar triangles, it may be shown that AT''' equals HT''; therefore, etc.

3. *The secant and cosecant of an arc are respectively equal to the secant and cosecant of the supplement of the arc.*

This may be demonstrated in a manner quite similar to those above. Let the pupil be required to show it.

4. If a equals any arc or angle, then we shall have, from the definitions,

$$\sin a = \cos\ (90^\circ - a)$$
$$\tan a = \cot\ (90^\circ - a)$$
$$\sec a = \operatorname{cosec} (90^\circ - a)$$

NATURAL SINES, COSINES, ETC.

21. The length of these trigonometrical lines may be expressed in numbers, differing, of course, as the radius of the circle is larger or smaller. If the radius is regarded as *unity*, or 1, we have what are called *natural sines, cosines*, etc. The method of calculating these sines, cosines, etc. will be explained hereafter.

The operation of multiplying and dividing by these natural sines being long and tedious, it has been found more convenient to use *logarithmic sines*, which we will now explain.

TABLE OF LOGARITHMIC SINES.

22. A LOGARITHMIC SINE, COSINE, TANGENT, or COTANGENT is the logarithm of the sine, cosine, tangent, or cotangent of an arc of a circle whose radius is 10,000,000,000.

23. A TABLE OF LOGARITHMIC SINES is a table containing the logarithmic sine, cosine, tangent, and cotangent of arcs.

24. The table of logarithmic sines may be calculated from a table of natural sines, as will be explained hereafter. In the table, the degrees are given at the top and bottom of the page, and the minutes at the sides, in the column headed M.

25. The column headed D contains the increase or decrease for 1 second. This is found by subtracting the logarithmic sine, etc. of an arc from that next exceeding it by 1 minute, and dividing the difference by 60.

26. *To find the logarithmic sines, cosines, etc. of arcs or angles.*

1. *When the arc is expressed in degrees, or in degrees and minutes.* If the angle is less than 45°, look for the degrees at the *top* of the page, and for the minutes in the *left*-hand column; then, opposite to the minutes, on the same horizontal line, in the column headed

Sine, will be found the logarithmic sine; in that headed *Cosine* will be found the logarithmic cosine, etc. Thus,

log sin 23° 35′	9.602150
log tan 23° 35′	9.640027

If the angle exceeds 45°, look for the degrees at the bottom of the page, and for the minutes in the *right*-hand column; then, opposite to the minutes, in the same horizontal line, in the column marked at the bottom *Sine*, will be found the logarithmic sine, etc. Thus,

log cos 65° 24′	9.619386
log tan 65° 24′	10.339290

2. *When the arc contains seconds.*—Find the logarithmic sine, etc. as before; then multiply the corresponding number found in column D by the number of seconds, and *add* the product to the preceding logarithm for the sines or tangents, and *subtract* it for cosines or cotangents.

We subtract for cosine and cotangent, because the greater the arc the less the cosine or cotangent. In multiplying the tabular difference by the number of seconds, we observe the same rule for the decimal point as in logarithms. If the arc is greater than 90°, we find the sine, cosine, etc. of its supplement.

EXAMPLES.

1. Find the logarithmic sine of 36° 24′ 42″.

SOLUTION.

log sin 36° 24′,		9.773361
Tabular difference,	2.85	
No. of seconds,	42	
Product,	119.70 to be added,	120
log sin 36° 24′ 42″,		9.773481

2. Find the logarithmic cosine of 64° 30′ 30″.

SOLUTION.

log cos 64° 30′,		9.633984
Tabular difference,	4.41	
No. of seconds,	30	
Product,	132.30 to be subtracted,	132
log cos 64° 30′ 30″,		9.633852

3. Find the logarithmic tangent of 120° 15′ 24″.

SOLUTION.

	180° 00′ 00″	
The given arc,	120 15 24	
Supplement,	59 44 36	
log tan 59° 44′,		10.233905
Tabular difference,	4.84	
No. of seconds,	36 to be added,	174.24
log tan 120° 15′ 24″,		10.234079

4. Find the logarithmic sine of 40° 40′ 40″. *Ans.* 9.814117.

5. Find the logarithmic cosine of 140° 30′ 20″.

Ans. 9.887441.

6. Find the logarithmic tangent of 85° 25′ 45″.

Ans. 11.097200.

7. Find the logarithmic cotangent of 144° 44′ 28″.

Ans. 10.150603.

27. *To find the arc corresponding to any logarithmic sine, cosine, tangent, or cotangent.*

1. Look in the proper column of the table for the given logarithm; if found there, and the name of the function be at the *head* of the column, take the degrees at the *top*, and the minutes on the *left;* but if the name of the function is at the *foot* of the column, take the degrees at the *bottom*, and the minutes on the *right.*

2. If the given logarithm is not exactly given in the table,

then take the next less logarithm, subtract it from the given logarithm, and divide the remainder by the corresponding tabular difference; the quotient will be seconds, which must be *added* to the degrees and minutes corresponding to the logarithm taken from the table, for *sines* and *tangents*, and *subtracted* for *cosines* and *cotangents*.

EXAMPLES.

1. Find the arc whose logarithmic sine is 9.617033.

SOLUTION.

Given log sine,		9.617033
Next less in table,		9.616894
Tabular difference,	4.63)	139.00(30, to be added.

Hence, the arc or angle is 24° 27′ 30″.

2. Find the arc whose logarithmic cosine is 9.704682.

SOLUTION.

Given log cosine,		9.704682
Next less in table,		9.704610
Tabular difference,	3.58)	72.00(20, to be subtracted.

Hence, the arc or angle is 59° 33′ 40″.

3. Find the arc whose logarithmic sine is 9.438672.

Ans. 15° 56′ 14″.

4. Find the arc whose logarithmic cosine is 9.634520. 59

Ans. 64° 27′ ~~47~~″.

5. Find the arc whose logarithmic tangent is 10.753246.

Ans. 79° 59′ 24″.

6. Find the arc whose logarithmic cotangent is 11.449852. 58″

Ans. 2° 1′ ~~40~~″.

28. Having learned how to find logarithmic sines, cosines, etc., we will next demonstrate some theorems for the solution of triangles.

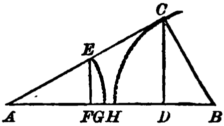
THE THEOREMS OF TRIGONOMETRY.

29. The Theorems of Trigonometry express the relation between the sides and trigonometrical functions of the angles of triangles.

30. We give five theorems, the first three relating to triangles in general, the others to right-angled triangles.

THEOREM I.

31. *In any plane triangle, the sides are proportional to the sines of the opposite angles.*



Let ABC be a plane triangle; then will

$$CB : CA :: \sin A : \sin B.$$

For, with A as a centre, and a radius AE equal to BC, describe the arc EG, and draw the perpendicular EF. With B as a centre, and the equal radius BC, describe the arc CH, and draw the perpendicular CD; then will CD be the sine of the angle B, and EF be the sine of the angle A, to the same radius. Now, by similar triangles (B. III. Th. X.),

$$AE : AC :: EF : CD.$$

But AE equals CB, EF is $\sin A$, and CD is $\sin B$.

Hence, $$BC : AC :: \sin A : \sin B.$$

In a similar manner, it may be shown that

$$AC : AB :: \sin B :: \sin C.$$

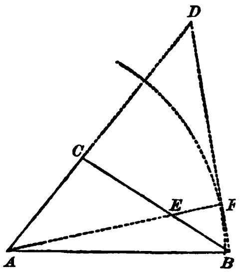
Therefore, etc.

THEOREM II.

32. *In any plane triangle, the sum of any two sides is to their difference as the tangent of half the sum of the opposite angles is to the tangent of half their difference.*

Let ABC be any plane angle; then will

$$BC + AC : BC - AC :: \tan \tfrac{1}{2}(A + B) : \tan \tfrac{1}{2}(A - B).$$



For, produce AC to D, making CD equal to CB, and draw BD; take CE equal to AC, draw AE, and produce it to F; then AD is the *sum* and BE the *difference* of the two sides AC and BC.

The sum of the angles CAE and AEC equals the sum of CAB and CBA, both sums being equal to 180° minus ACB (B. I. Th. XIII.); but the angle CAE equals AEC (B. I. Th. X.); hence, CAE or CAF is the *half sum* of CAB and CBA; also, BAF is the *half difference* of the angles CAB and ABC, since it equals the *half sum* CAE, subtracted from the greater angle CAB.*

The angle CDF equals CBD, since CB equals CD; also, CAE, which equals AEC, is equal to the vertical angle FEB; hence, the third angles of the triangles, AFD and EFB, are equal, and, therefore, AF is perpendicular to BD; consequently, if then we regard AF as the radius, FD will be the tangent of DAF, and FB will be the tangent of FAB. Now, by similar triangles,

$$AD : EB :: FD : FB; \text{ or,}$$

$$CB + AC : CB - AC :: \tan \tfrac{1}{2}(A + B) : \tan \tfrac{1}{2}(A - B).$$

THEOREM III.

33. *In any plane triangle, if a line is drawn from the vertical angle perpendicular to the base, then the whole base will be to the sum of the other two sides as the difference of those sides is to the difference of the segments of the base.*

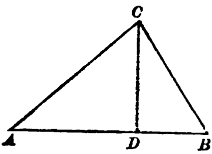
Let ABC be a triangle, and CD perpendicular to the base; then will

$$AB : AC + BC :: AC - BC : AD - DB.$$

---

* This principle is thus proven:—Let a and b be any two quantities; then the half sum is $\frac{a+b}{2}$, and the half difference is $\frac{a-b}{2}$; and $a - \frac{a+b}{2} = \frac{a-b}{2}$; that is, *the greater minus the half sum equals the half difference.*

For, from Th. VI. Book III.,



$$\overline{AC}^2 = \overline{AD}^2 + \overline{DC}^2,$$

and $$\overline{BC}^2 = \overline{BD}^2 + \overline{DC}^2.$$

Subtracting, $\overline{AC}^2 - \overline{BC}^2 = \overline{AD}^2 - \overline{BD}^2.$

Hence (B. III. Th. V. C. 2),

$$(AC+BC)\times(AC-BC)=(AD+BD)\times(AD-BD);$$

therefore, $AD+DB : AC+BC :: AC-BC : AD-DB.$

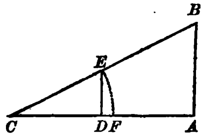
Therefore, etc.

THEOREM IV.

34. *In any right-angled plane triangle, radius is to the sine of either angle as the hypothenuse is to the side opposite.*

Let CAB be a triangle right-angled at A, and denote the radius by R: then will

$$R : \sin C :: CB : AB.$$



For, from the point C as a centre and any radius, as CE, describe the arc EF, and draw ED perpendicular to CA; then will ED be the sine of the angle C. The two triangles CED and CAB are similar; hence, we have (B. III. Th. X.),

$$CE : ED :: CB : BA,$$

or, $$R : \sin C :: CB : BA.$$

Therefore, etc.

Cor. It may also be shown that *radius is to the cosine of either acute angle as the hypothenuse is to the side adjacent.*

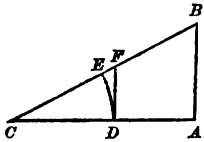
THEOREM V.

35. *In any right-angled plane triangle, radius is to the tangent of either acute angle as the side adjacent is to the side opposite.*

Let CAB be a triangle right-angled at A; then will

$$R : \tan C :: CA : AB.$$

For, with C as a centre and any radius CD, describe the arc



DE, and draw *DF* perpendicular to *CA*; *FD* will be the tangent of the angle *C*. The triangles *CDF* and *CAB* are similar; hence,

$$CD : DF :: CA : AB,$$

or,

$$R : \tan C :: CA : AB.$$

Therefore, etc.

Cor. It may also be shown that *radius is to cotangent of either angle, as side opposite is to side adjacent.*

SOLUTION OF TRIANGLES.

36. The Solution of a Triangle is the process of finding the unknown parts when a sufficient number of the parts are given.

37. There are *six parts* in a plane triangle, and three of these—one of the three being a side—must be given to find the other parts.

38. If the angles alone were given, it is clear that the sides could not be determined, since there could be an indefinite number of triangles having their angles respectively equal.

39. There are four cases, as follows:

1. When two angles and a side are given.
2. When two sides and an angle are given.
3. When two sides and the included angle are given.
4. When the three sides are given.

CASE I.

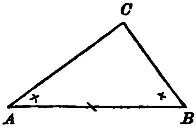
40. *Given two angles and one side, to find the remaining parts.*

Method.—We subtract the sum of the given angles from 180° to find the third angle, and then find the sides by Theorem I.

EXAMPLES.

1. In a triangle *ABC*, there are given the angle $A = 32° 24'$, the angle $B = 40° 32'$, and the side $AB = 240$; required the other parts.

SOLUTION.—Let ABC represent the triangle; then the sum of A and $B = 72° 56'$, and $C = 180° - 72° 56' = 107° 04'$. Then, to find AC, we have,



$$AC : AB :: \sin B : \sin C.$$

Hence, $$AC = AB \times \sin B \div \sin C.$$

From which AC is readily found by multiplying 240 by the natural sine of B, and dividing by the natural sine of C. It is simpler, however, to use logarithms. To find AC, we add the log of AB and log sin B, and subtract log sin C, or *add* the *arith. comp.* of log sin C.

a. c. log sin C	(107° 04′),	0.019558
log sin B	(40° 32′),	9.812840
log AB	(240),	2.380211
log AC;		2.212609 $\therefore AC = 163.158$

To find the side BC, we have,

$$BC : AB :: \sin A : \sin C:$$

or, by logarithms,

a. c. log sin C	(107° 04′),	0.019558
log sin A	(32° 24′),	9.729024
log AB	(240),	2.380211
log BC,		2.128793 $\therefore BC = 134.522$

2. In the triangle ABC, there are given the angle $A = 27° 40'$, the angle $C = 65° 45'$, and the side $AB = 625$, to find the other parts. *Ans.* $B = 86° 35'$; $BC = 318.29$; $AC = 684.266$.

CASE II.

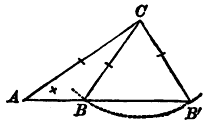
41. *Given two sides and an angle opposite one of them, to find the remaining parts.*

METHOD.—One of the required angles is found by Theorem I. The third angle is found by subtracting the sum of the two from 180°; the third side is found by Case I.

EXAMPLES.

1. In the triangle ABC, there are given $AC = 200$, $CB = 150$, and the angle $A = 44° 26'$, to find the other parts.

SOLUTION.—Let ABC be a triangle in which $A = 44° 26'$, $AC = 200$, and $BC = 150$; then, to find the angle B, we have,



$$\sin B : \sin A :: AC : BC,$$

or,	BC	(150)	a. c.	7.823009
	$: AC$	(200)		2.301030
	$:: \sin A$	(44° 26′)		9.845147
	$: \sin B$	( )		9.969186

$$\therefore B = 68° 40' 16'', \text{ or, } 111° 19' 44''$$

In this problem, if the side BC, opposite the given angle A, is shorter than the other given side AC, the solution will be *ambiguous;* for *two triangles*, ACB and ACB', may be formed, each of which will satisfy the conditions of the problem. Hence, the angle B found above may be either ABC or B'. But these, it will be seen, are supplements of each other; hence, in finding the angle corresponding to sin B, we take the angle or its supplement.

In practice, there is often some circumstance to determine whether the angle is acute or obtuse. *If the angle given is obtuse,* the other angles must be acute, and there will be but one solution. *If the side BC is equal to or greater than AC,* there will be but one triangle.

In the given diagram above, the angle $ABC = 111° 19' 44''$, and $AB'C = 68° 40' 16''$; hence, the angle $ACB = 24° 13' 16''$, and the angle $ACB' = 113° 6' 16''$.

To find the side AB, we have,

$$AB : CB :: \sin ACB : \sin A;$$

from which, by logarithms, we find $AB = 88.085$.

To find the side AB', we have,

$$AB' : CB' :: \sin ACB' : \sin A;$$

from which, by logarithms, we find $AB' = 197.484$.

2. In a triangle ABC there are given AB 45.96, BC 62.50, and the angle A 79° 21′; find the remaining parts.

Ans. $C = 46° 16' 38''$; $B = 54° 22' 22''$; $AC = 51.69$.

(There is no ambiguity, since the side BC is greater than AC.)

3. In a triangle ABC there are given $BC = 15.71$, $AC = 21.12$, and the angle $A = 27° 50'$; find the other parts.

Ans. $C = 113° 17' 13''$; $B = 38° 52' 47''$; $AB = 30.906$.
or, $C = 11° 2' 47''$; $B = 141° 7' 13''$; $AB = 6.447$.

CASE III.

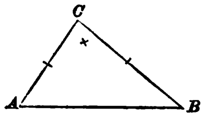
42. *Given two sides and the included angle, to find the remaining parts.*

METHOD.—We find the sum of the two angles by subtracting the given angle from 180°, and divide this by 2 for the *half sum.* We then find the *half difference*, by Theorem II. Having found the half sum and half difference of the two angles, we find the greater angle by adding the half difference to the half sum; and the less by subtracting the half difference from the half sum. The third side is found by Theorem I.

EXAMPLES.

1. In the triangle ABC, let $BC = 680$, $AC = 460$, and the included angle 84°; required the other parts.

SOLUTION.—Let ABC represent the triangle, $AC = 460$, $BC = 680$, and the angle $C = 84°$. Then, $AC + BC = 460 + 680 = 1140$; $BC - AC = 680 - 460 = 220$. $A + B = 180° - 84° = 96°$; hence, *half sum* $= 48°$. The half difference we find by the following proportion.



$BC + AC$	1140	ar. co.	6.943095
: $BC - AC$	220	. .	2.342423
:: $\tan \frac{1}{2} (A + B)$	48°	. .	10.045563
: $\tan \frac{1}{2} (A - B)$	12° 5′ 49″		9.331081

Hence, $A =$ 60° 5′ 49″; and $B = 35° 54' 11''$.

The other side, found by Theorem I., equals 783.733.

2. Given two sides of a plane triangle 240 and 360, and the included angle 68° 36′; required the other parts.

Ans. 72° 02′ 26; 39° 21′ 34″; 352.349.

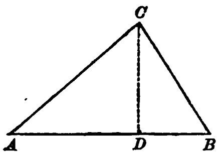
CASE IV.

43. *Given the three sides of a plane triangle, to find the angles.*

METHOD.—Let fall a perpendicular upon the greater side from the angle opposite, dividing the triangle into two right-angled triangles. Find the difference of the segments of the base by Theorem III.; half this difference added to half the base gives the greater segment, and subtracted from half the base gives the less. We will then have two sides and the right angle of two right-angled triangles, from which we can find the acute angles by Theorem I.

EXAMPLES.

1. In a triangle ABC, given $AB = 60$, $AC = 50$, and $BC = 40$, to find the angles.



SOLUTION.—Let ABC represent the triangle; then $AB = 60$, $AC = 50$, $BC = 40$; then, by Th. III.,

$$AB : AC + BC :: AC - BC : AD - BD,$$

or, $60 : \quad 90 \quad :: \quad 10 \quad : AD - BD.$

hence, $\quad AD - BD = 90 \times 10 \div 60 = 15;$

then, $\quad AD = \frac{1}{2}(60 + 15) = 37.5$

and $\quad BD = \frac{1}{2}(60 - 15) = 22.5$

Then, in the triangle ACD, to find the angle ACD,

a. c.	AC	(50)	8.301030
	$: AD$	(37.5)	1.574031
	$:: \sin D$	(90°)	10.000000
	$: \sin ACD$	48° 35′ 25″	9.875061

Then, in the triangle BCD, to find the angle BCD,

a. c.	BC	(40)	8.397940
	$: BD$	(22.5)	1.352183
	$:: \sin D$	(90°)	10.000000
	$: \sin BCD$	34° 13′ 44″	9.750123

Hence, $A = 90° - 48° \ 35' \ 25'' = 41° \ 24' \ 35''$,

and $B = 90° - 34° \ 13' \ 44'' = 55° \ 46' \ 16''$,

and $C = 48° \ 35' \ 25'' + 34° \ 13' \ 44'' = 82° \ 49' \ 09''$.

2. In a plane triangle the sides are 1005, 1210, and 1368; required the angles.

Ans. 45° 22′ 35″; 58° 58′ 18″; 75° 39′ 7″.

SOLUTION OF RIGHT-ANGLED TRIANGLES.

44. In the solution of right-angled triangles we have the four following cases:

1. When the hypothenuse and an acute angle are given.
2. When the hypothenuse and a side are given.
3. When one side and the angles are given.
4. When the two sides about the right angle are given.

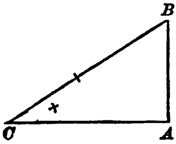
Method.—The first three cases are readily solved by Theorem I.; remembering that the sine of 90° is *radius*, the log. sin. being 10. The fourth case may be solved by Theorem V.; or we may find the hypothenuse by B. III. Th. VI., and then find the angles by Theorem I.

These four cases may also be solved by Theorems IV. and V.; but the method suggested above is preferred, since it is simpler and more easily remembered.

EXAMPLES.

1. In a right-angled triangle, given the hypothenuse 475 and the angle at the base 36° 34′; find the other parts.

SOLUTION.—Let CAB represent the triangle, BC being equal to 475 and the angle $C = 36° \; 34'$; then, to find AB, we have,



$\sin A$	$90°$	a. c.	0.000000
$: \sin C$	$36° \; 34'$		9.775070
$:: CB$	475		2.676694
$: AB$	282.985		2.451764

The angle $B = 90° - (36° \; 34') = 53° \; 26'$; then, by a similar proportion, we can find the side $CA = 381.503$.

2. Given the hypothenuse 45.36 and the angle at the base 45° 36′; required the other parts. *Ans.*

3. Given the hypothenuse 396 and the base 218, to find the other parts. *Ans.* 330.59; 33° 24′ 05″; 56° 35′ 55″.

4. Given the two sides 58.75 and 74.58, to find the remaining parts. *Ans.* 94.94; 38° 13′ 45″; 51° 46′ 15″.

PRACTICAL APPLICATIONS.

HEIGHTS AND DISTANCES.

45. A HORIZONTAL PLANE is one which is parallel to the plane of the horizon.

46. A VERTICAL PLANE is one which is perpendicular to a horizontal plane.

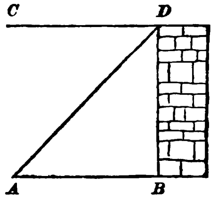
47. A HORIZONTAL LINE is any line in a horizontal plane. A *vertical line* is a line perpendicular to a horizontal plane.

48. A HORIZONTAL ANGLE is an angle in a horizontal plane.

49. A VERTICAL ANGLE is an angle in a vertical plane.

50. An ANGLE OF ELEVATION is a vertical angle having one

side horizontal, and the inclined side above the horizontal side; as *BAD*.



51. An Angle of Depression is a vertical angle having one side horizontal, and the inclined side under the horizontal side; as *CDA*.

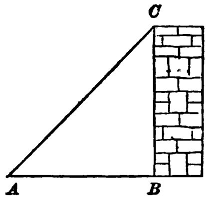
52. *Distances upon the ground* are usually measured by a chain, called *Gunter's Chain*. This chain is 4 rods or 66 feet long, and consists of 100 links. Sometimes a half chain is used, consisting of 50 links.

53. Angles are measured by various instruments. Horizontal angles are measured by an instrument called *The Compass*. Horizontal and vertical angles are both measured by the *Theodolite*, or, what is still better for general use, a *Transit-Theodolite*.

CASE I.

54. *To determine the height of a vertical object standing upon a horizontal plane.*

Method.—Measure from the foot of the object any convenient horizontal distance *AB*; at the point *A* take the angle of elevation *BAC*; then, in the triangle *ABC* we have a side and an acute angle; hence, we can readily find the altitude.



1. From the foot of a tower I measure a horizontal line 120 feet, and at its extremity find the angle of elevation to be 48° 36′; what was the height of the tower?

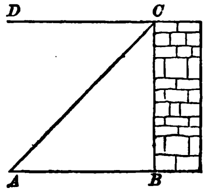
Ans. 136.113 feet.

CASE II.

55. *To find the distance of a vertical object whose height is known.*

Method.—Measure the angle of elevation to the top of the object, as before; we will then have a right-angled triangle in

which we know the perpendicular and an acute angle; hence, we can readily find the base.



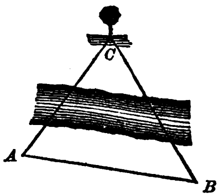
1. I took the angle of elevation to the top of a flag-staff whose height I knew to be 160 feet, and found it to be 20°; how far was I from the staff?

Ans. 439.60 feet.

CASE III.

56. *To find the distance of an inaccessible object.*

METHOD.—Measure a horizontal base-line AB, and then take the angles formed by this line and lines from the object to the extremities of this base-line, as CAB and ABC; the distance AC or BC can then be readily found.



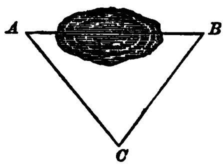
1. I am on one side of a river, and wish to know the distance to a tree on the other side. I measure 300 yards by the side of the river, and find that the two angles formed by this line and the lines from its extremities to the tree are 72° 40′ and 45° 36′; required the distance from each extremity of the base-line to the tree.

Ans. 243.362 yards; 325.15 yards.

CASE IV.

57. *To find the distance between two objects separated by an impassable barrier.*

METHOD.—Select any convenient station, as C, and measure the distance from it to each of the objects A and B, and the angle C included between these lines. We can then readily find the distance AB.



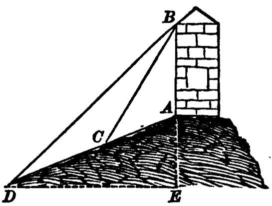
1. The distance between two trees cannot be directly measured: I therefore take a third position from

which each of the trees can be seen, and find the distances from it to the trees to be 300 and 250 yards, and the included angle 43° 16′; required the distance between the trees.

Ans. 208.02 yards.

CASE V.

58. *To find the height of a vertical object standing upon an inclined plane.*



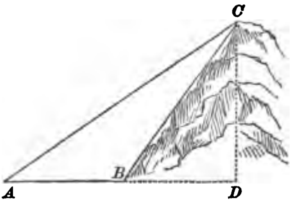
METHOD.—Measure any convenient distance DA on a line from the foot of the object, and at the point D measure the angles of elevation, EDA and EDB, to foot and top of the tower. By means of the two triangles DEA and DEB, we can find the height of AB.

1. Wishing to determine the height of a tower situated upon a hill, I measured a distance down the slope of the hill 400 feet, and found the angles of elevation to the foot of the tower 42° 28′, and to the top of the tower 68° 42′; required the height of the tower.

Ans. 486.747.

CASE VI.

59. *To find the height of an inaccessible object above a horizontal plane.*

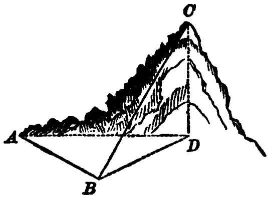


FIRST METHOD.—Measure any convenient horizontal line AB directly toward the object, and take the angles of elevation at A and B; we will then have conditions sufficient to find DC.

1. Wishing to find the altitude of a hill, I measured the angle of elevation at the bottom 60° 37′, and 460 feet from the foot in a right line of the top of the hill and the point at the foot, and in the same horizontal plane as the foot, I measured the angle of elevation 36° 52′; required the height of the hill.

Ans. 597.092.

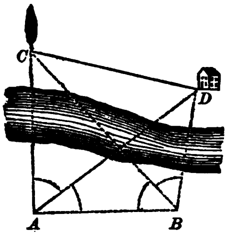
SECOND METHOD.—*If it is not convenient to measure a horizontal base-*



line towards the object, we measure any line AB, and also measure the horizontal angles BAD, ABD, and the angle of elevation DBC. Then, by means of the two triangles ABD and CBD, the height CD can be found.

CASE VII.

60. *To find the distance between two inaccessible objects when points can be found at which both objects can be seen.*



METHOD.—The method of measurement is indicated in the following problem. The method of solution we prefer leaving to the ingenuity of the pupil, that he may learn to think for himself.

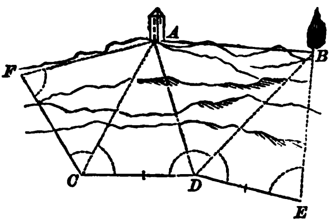
1. Wishing to know the horizontal distance between a tree and house on the opposite side of a river, I took the following measurements:

$$AB = 400; \quad CAD = 56° \; 30',$$
$$BAD = 42° \; 24'; \; ABC = 44° \; 36',$$
$$\text{and } DBC = 68° \; 50'.$$

Required the distance CD. *Ans.* 747.913.

CASE VIII.

61. *To find the distance between two inaccessible objects when no points can be found from which both objects can be seen.*



METHOD.—The method is indicated in the following problem and figure. This and the following case may be omitted with young pupils.

1. Wishing to know the horizontal distance between two in-

accessible objects when no point can be found from which both objects can be seen, two objects C and D are taken, 600 feet apart, from the former of which A can be seen, from the latter B. From C we measure the distance CF, not in the direction DC, equal to 600 feet, and from D a distance DE equal to 600 feet. We then measure the following angles:

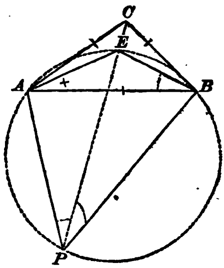
$$CFA = 80^\circ\ 16', \quad BED = 86^\circ\ 25',$$
$$ACF = 52^\circ\ 24', \quad BDE = 60^\circ\ 24',$$
$$ACD = 56^\circ\ 36', \quad BDC = 150^\circ\ 30'.$$

Required the distance AB. *Ans.* 1117.44 feet.

CASE IX.

62. *To find the distances from a given point to three objects whose distances from each other are known.*

METHOD.—The method is indicated in the problem and figure.



1. I wish to locate three buoys, A, B, and C, in a harbor, so that the distance between A and B is 800 yards, between A and C 600 yards, between B and C 400 yards, and from a fixed point on shore, the angle APC shall equal $33^\circ\ 45'$, and BPC $22^\circ\ 30'$; required the distances PA, PC, and PB.

Ans. $PA = 710.193$; $PC = 1042.522$; $PB = 934.291$.

NOTE.—This last problem is given by quite a number of authors, and seems to be general property.

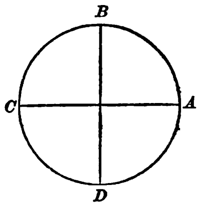
ANALYTICAL TRIGONOMETRY.

63. ANALYTICAL TRIGONOMETRY is that branch of Mathematics which treats of the properties and relations of trigonometrical functions.

64. Trigonometry, in its origin, was confined to triangles, the method of reasoning being geometrical. After the invention of *analysis*, mathematicians began to apply it to trigonometry, and, in course of time, developed the general properties of trigonometrical functions. This has enlarged the science and greatly increased its power as an instrument of investigation and discovery.

DEFINITIONS.

65. A circumference consists of four *quadrants*. AB is the *first quadrant;* BC is the *second quadrant,* etc.



66. The *origin* of arcs is at A, all arcs being generally supposed to begin at A.

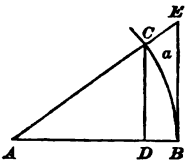
67. The *extremity* of an arc is where it ends. An arc is said to be in that quadrant where its extremity is situated.

68. The sine, cosine, tangent, cotangent, etc. of an arc have already been defined, and need not be repeated here. The *versed sine* of an arc is the distance from the foot of the sine to the origin of the arc. The *co-versed sine* is the versed sine of the complement.

The sines, cosines, etc. are called the *circular functions* of the arcs.

69. FUNDAMENTAL FORMULAS EXPRESSING THE RELATION BETWEEN THE CIRCULAR FUNCTIONS OF ANY ARC.

1. Let a represent the measuring arc of any angle. Draw the lines represented in the figure. Then, from the definitions,



$AB = 1$, $BE = \tan a$,

$CD = \sin a$, $AE = \sec a$,

$AD = \cos a$, $DB = \text{ver sin } a$.

In the right-angled triangle ADC, we have,

$\overline{CD}^2 + \overline{AD}^2 = AC^2$, or, by substitution,

$\sin^2 a + \cos^2 a = 1$. (1)

Hence, $\sin^2 a = 1 - \cos^2 a$; (2) $\cos^2 a = 1 - \sin^2 a$. (3)

2. From the figure, we also have,

$$DB = AB - AD; \text{ that is,}$$

$$\text{ver sin } a = 1 - \cos a. \tag{4}$$

Since this is true for any value of a, it is true for $90° - a$;

hence, $\text{ver sin } (90° - a) = 1 - \cos (90° - a)$,

or, $\text{co-ver sin } a = 1 - \sin a$. (5)

3. Again, the triangles ADC and ABE being similar,

$$EB : AB :: CD : AD,$$

or, $\tan a : 1 :: \sin a : \cos a$;

hence, $\tan a = \dfrac{\sin a}{\cos a}$. (6)

Substituting $90° - a$ for a, we have,

$$\tan (90° - a) = \frac{\sin (90° - a)}{\cos (90° - a)},$$

or, $\cot a = \dfrac{\cos a}{\sin a}$. (7)

4. Again, multiplying equations (6) and (7), we have,

$$\tan a \cot a = 1; \tag{8}$$

hence, $\tan a = \dfrac{1}{\cot a}$ (9), and $\cot a = \dfrac{1}{\tan a}$. (10)

5. Again, from the same triangles, we have,

$$AE : AB :: AC : AD,$$

or, $\sec a : 1 :: 1 : \cos a$;

hence, $\sec a = \dfrac{1}{\cos a}$. (11)

Substituting $90° - a$ for a,

$$\sec (90° - a) = \frac{1}{\cos (90° - a)},$$

or, $\text{cosec } a = \dfrac{1}{\sin a}$. (12)

6. Again, from the triangle ABE, we have,

$$\sec^2 a = 1 + \tan^2 a; \tag{13}$$

hence,

$$\operatorname{cosec}^2 a = 1 + \cot^2 a. \tag{14}$$

70. These are the fundamental formulas of trigonometry, and should be committed to memory. We will collect them, forming the following table:

TABLE I.

1. $\operatorname{Sin}^2 a + \cos^2 a$	=	1	9. $\operatorname{Tan} a$	=	$\dfrac{1}{\cot a}$
2. $\operatorname{Sin}^2 a$	=	$1 - \cos^2 a$			
3. $\operatorname{Cos}^2 a$	=	$1 - \sin^2 a$	10. $\operatorname{Cot} a$	=	$\dfrac{1}{\tan a}$
4. $\operatorname{Ver\,sin} a$	=	$1 - \cos a$			
5. $\operatorname{Co\text{-}ver\,sin} a$	=	$1 - \sin a$	11. $\operatorname{Sec} a$	=	$\dfrac{1}{\cos a}$
6. $\operatorname{Tan} a$	=	$\dfrac{\sin a}{\cos a}$	12. $\operatorname{Co\text{-}sec} a$	=	$\dfrac{1}{\sin a}$
7. $\operatorname{Cot} a$	=	$\dfrac{\cos a}{\sin a}$	13. $\operatorname{Sec}^2 a$	=	$1 + \tan^2 a$
8. $\operatorname{Tan} a \cot a$	=	1	14. $\operatorname{Co\text{-}sec}^2 a$	=	$1 + \cot a$

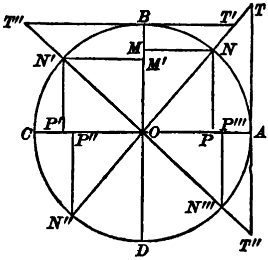
ALGEBRAIC SIGNS OF THE CIRCULAR FUNCTIONS.

71. In analytical trigonometry, we regard the *algebraic signs* of the functions as well as their numerical value. The sign of a function is determined by the following principles.

1. *All lines estimated upward from the horizontal diameter are* POSITIVE; *all lines estimated downward from it are* NEGATIVE.

2. *All lines estimated from the vertical diameter towards the right are* POSITIVE; *all lines estimated toward the left are* NEGATIVE.

Thus, the sines NP and $N'P'$ are positive, while $N''P''$ and $N'''P'''$ are negative; so also the cosines OP and OP''' are positive, while OP' and OP'' are negative.



72. The simplest way to determine the algebraic signs of the different functions is to derive those of the sine and cosine from the figure, and the others from the formulas.

1. The SINE is *positive* in the *first* and *second* quadrants, being measured above, and *negative* in the *third* and *fourth* quadrants.

2. The COSINE is *positive* in the *first* and *fourth* quadrants, and *negative* in the *second* and *third* quadrants.

3. The TANGENT is *positive* in the *first* and *third* quadrants, and *negative* in the *second* and *fourth.*

For, from formula (6),

$$\tan a = \frac{\sin a}{\cos a},$$

and this is positive when sine and cosine have like signs, and negative when they have unlike signs. In the first quadrant, both sine and cosine are plus, in the third both are minus, in the second and fourth one is plus and the other minus; hence, the tangent is positive in the first and third quadrants and negative in the second and fourth.

4. The COTANGENT is *positive* in the *first* and *third* quadrants, and *negative* in the *second* and *fourth;* as is readily shown from the formula,

$$\cot a = \frac{\cos a}{\sin a}.$$

5. The SECANT is *positive* in the *first* and *fourth* quadrants, and *negative* in the *second* and *third.* For, from formula (11),

$$\sec a = \frac{1}{\cos a};$$

hence, the secant has the same sign as the cosine.

6. The CO-SECANT is *positive* in the *first* and *second* quadrants, and *negative* in the *third* and *fourth,* as may be shown from For. (12).

NOTE.—Some of these may also be readily shown from the figure. In the secant, when the distance is estimated *toward* the extremity of the arc, it is *plus;* when *from* the extremity, *minus.*

LIMITING VALUES OF THE CIRCULAR FUNCTIONS.

73. The *limiting values* of the circular functions are their values at the beginning and end of the different quadrants.

These values are determined by the principle that *the value of a variable quantity up to the limit is its value at the limit.*

Beginning at the *origin*, we see that the value of sin 0 is 0, and the cos 0 is the radius, or 1. As the arc increases, the sine increases and the cosine decreases, until at 90° the sine is 1 and the cosine 0. As the arc increases from 90° to 180°, the sine decreases and cosine increases *numerically* (diminishes algebraically), until at 180° the sine is $+0$ and cosine -1. In the same way we see that $\sin 270° = -1$, and $\cos 270° = -0$; also, $\sin 360° = -0$, and cosine $360° = 1$.

Now, since, by formula (6),

$$\tan a = \frac{\sin a}{\cos a},$$

substituting 0 for a, $$\tan 0 = \frac{\sin 0}{\cos 0} = \frac{0}{1} = 0;$$

and, also, $$\cot 0 = \frac{\cos 0}{\sin 0} = \frac{1}{0} = \infty.$$

74. By a similar examination of the limiting values of all the functions, we have the following table:

TABLE II.

Arc = 0		Arc = 90°		Arc = 180°		Arc = 270°		Arc = 360°	
sin =	0	sin =	1	sin =	0	sin =	-1	sin =	-0
cos =	1	cos =	0	cos =	-1	cos =	-0	cos =	1
v-sin =	0	v-sin =	1	v-sin =	2	v-sin =	1	v-sin =	0
co-v-sin =	1	co-v-sin =	0	co-v-sin =	1	co-v-sin =	2	co-v-sin =	1
tan =	0	tan =	∞	tan =	-0	tan =	∞	tan =	-0
cot =	∞	cot =	0	cot =	$-\infty$	cot =	0	cot =	$-\infty$
sec =	1	sec =	∞	sec =	-1	sec =	$-\infty$	sec =	1
cosec =	∞	cosec =	1	cosec =	∞	cosec =	-1	cosec =	$-\infty$

FUNCTIONS OF THE SUM OR DIFFERENCE OF AN ARC AND ANY NUMBER OF QUADRANTS.

75. The trigonometrical function of any arc formed by adding an arc to or subtracting it from any number of quadrants, may be expressed in functions of the arc which is added to or subtracted from.

1. Let a represent any arc less than 90°; then, from the definitions, we have,

$$\sin(90^\circ - a) = \cos a, \qquad \cot(90^\circ - a) = \tan a,$$
$$\cos(90^\circ - a) = \sin a, \qquad \sec(90^\circ - a) = \operatorname{cosec} a,$$
$$\tan(90^\circ - a) = \cot a, \qquad \operatorname{cosec}(90^\circ - a) = \sec a.$$

2. Now, let a represent the arc BN', then will $ABN' = 90^\circ + a$. From the figure, Art. 71, we see that

$$N'M' = \sin a, \qquad M'O = \cos a,$$
$$P'O = \cos(90^\circ + a), \qquad N'P' = \sin(90^\circ + a).$$

Hence, remembering that ABN', being in the second quadrant, its cosine is negative, we have,

$$\sin(90^\circ + a) = \cos a, \text{ and } \cos(90^\circ + a) = -\sin a.$$

Substituting these values in the formulas for tan, cot, etc. found in Table I., we have,

$$\tan(90^\circ + a) = -\cot a, \qquad \sec(90^\circ + a) = -\operatorname{cosec} a,$$
$$\cot(90^\circ + a) = -\tan a, \qquad \operatorname{cosec}(90^\circ + a) = \sec a.$$

3. Again, let a represent the arc CN', then will $ABN' = 180 - a$. From the figure, we have,

$$N'P' = \sin a, \qquad P'O = \cos a,$$
$$N'P' = \sin(180 - a), \qquad P'O = \cos(180 - a).$$

Hence, remembering that the cosine of ABN' ending in the second quadrant is negative, we have,

$$\sin(180^\circ - a) = \sin a, \text{ and } \cos(180^\circ - a) = -\cos a.$$

Substituting these values in the formulas for tan, cot, etc. in Table I., we have,

$$\tan(180^\circ - a) = -\tan a, \qquad \sec(180^\circ - a) = -\sec a,$$
$$\cot(180^\circ - a) = -\cot a, \qquad \operatorname{cosec}(180^\circ - a) = \operatorname{cosec} a.$$

From the above, we see that *the sine of an arc equals the sine of its supplement, and the cosine of an arc equals minus the cosine of its supplement*, etc.

76. In a similar manner, by deriving the values of the sines and cosines from the figure and making the substitutions in the proper formulas, we may obtain the functions of $180^\circ + a$, $270^\circ - a$, $270^\circ + a$, and $360^\circ - a$. All of these, with the above, are exhibited in the following table:

TABLE III.

Arc $= 90° + a$.		Arc $= 270° - a$.	
$\sin = \cos a$,	$\cot = -\tan a$,	$\sin = -\cos a$,	$\cot = \tan a$,
$\cos = -\sin a$,	$\sec = -\text{cosec}\, a$,	$\cos = -\sin a$,	$\sec = -\text{cosec}\, a$,
$\tan = -\cot a$,	$\text{cosec} = \sec a$.	$\tan = \cot a$,	$\text{cosec} = -\sec a$.
Arc $= 180° - a$.		**Arc $= 270° + a$.**	
$\sin = \sin a$,	$\cot = -\cot a$,	$\sin = -\cos a$,	$\cot = -\tan a$,
$\cos = -\cos a$,	$\sec = -\sec a$,	$\cos = \sin a$,	$\sec = \text{cosec}\, a$,
$\tan = -\tan a$,	$\text{cosec} = \text{cosec}\, a$.	$\tan = -\cot a$,	$\text{cosec} = -\sec a$.
Arc $= 180° + a$.		**Arc $= 360° - a$.**	
$\sin = -\sin a$,	$\cot = \cot a$,	$\sin = -\sin a$,	$\cot = -\cot a$,
$\cos = -\cos a$,	$\sec = -\sec a$,	$\cos = \cos a$,	$\sec = \sec a$,
$\tan = \tan a$,	$\text{cosec} = -\text{cosec}\, a$.	$\tan = -\tan a$,	$\text{cosec} = -\text{cosec}\, a$.

77. This table can easily be committed to memory, by observing that when the arc is connected with 180° or 360°, the functions in both columns have the *same name;* but when connected with 90° or 270°, the functions in the two columns have *different names.*

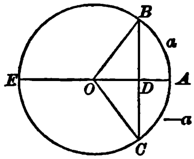
78. The principles of this table are of great value. By their means *the functions of any arc may be expressed in functions of an arc less than* 90°. Thus,

$$\sin\ 120° = \sin\ (90° + 30°) = \cos\ 30°,$$
$$\tan\ 243° = \tan\ (180° + 63°) = \tan\ 63°,$$
$$\cot\ 304° = \cot\ (270° + 34°) = -\tan\ 34°.$$

79. When the arc is greater than 360°, we may subtract 360° one or more times until we obtain an arc less than 360°; the *remainder* will have the same *origin* and *extremity:* hence, the circular function of the remainder will be the same as of the given arc, and this remainder being less than 360°, its functions can be expressed in functions of an arc less than 90°. Hence, *the functions of any arc can be expressed in functions of an arc less than* 90°.

CIRCULAR FUNCTIONS OF NEGATIVE ARCS.

80. Suppose AB to be any arc, and AC, estimated from the origin downward, be numerically equal to AB; then, if the arc AB be denoted by a, the arc AC will be denoted by $-a$; and CD will be the sine, and OD the cosine, of $-a$.



Now, since $BD = CD$ and OD is the cosine of both a and $-a$, we have,

$$\sin(-a) = -\sin a, \text{ and } \cos(-a) = \cos a.$$

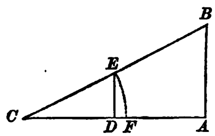
Substituting these in the formulas of Table I., we will have,

$$\begin{array}{llll} \text{ver sin} & (-a) = \text{ver sin } a, & \cot & (-a) = -\cot a, \\ \text{co-ver sin} & (-a) = 1 + \sin a, & \sec & (-a) = \sec a, \\ \tan & (-a) = -\tan a, & \text{co-sec} & (-a) = -\text{co-sec } a. \end{array}$$

81. From what has now been presented, we see that the circular functions of all arcs, whether positive or negative, may be expressed in functions of arcs less than 90°; hence, in the *tables of sines, cosines*, etc., we have only positive arcs and those less than 90°.

RELATION OF THE SIDES AND FUNCTIONS OF RIGHT-ANGLED TRIANGLES.

82. Let ACB be a right-angled triangle, the right angle being at A. Represent the angles by A, B, C, and their opposite sides by a, b, c. With a radius $CE = 1$, describe the arc EF, and draw the perpendicular ED; then $ED = \sin C$, and $CD = \cos C$.



Now, from the figure, we readily obtain,

$$1 : \sin C :: a : c,$$

and, also,

$$1 : \cos C :: a : b;$$

hence, $\quad \sin C = \frac{c}{a} \quad (1), \quad \cos C = \frac{b}{a} \quad (2),$

or, $\quad c = a \sin C \ (3), \quad b = a \cos C \ (4).$

Dividing (1) by (2) and then (2) by (1), we have,

$$\tan C = \frac{c}{b} \quad (5), \ \cot C = \frac{b}{c} \quad (6),$$

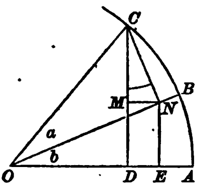
or, $$c = b \tan C \, (7), \text{ and } b = c \cot C \ (8).$$

83. These the pupil will commit to memory, and also translate into common language. The first, thus translated, is as follows:

1. *The sine of either acute angle of a right-angled triangle is equal to the opposite side divided by the hypothenuse.*

84. **General formulas relating to the sum and difference of arcs, double arcs, etc.**

1. Let AB and BC be two arcs having the common radius OA or $OC = 1$; denote AB by b and BC by a. From C draw CD perpendicular to OA, and CN perpendicular to OB; from N draw NE perpendicular to OA, and NM parallel to OA. Then,



$CD = \sin(a+b)$, $CN = \sin a$, $ON = \cos a$.

Now, $$CD = CM + NE.$$

In the triangle OEN,

$$NE = ON \sin B = \cos a \sin b;$$

since CMN and NOE are similar, and the angle $MCN = NOE = b$,

$$CM = CN \cos b = \sin a \cos b.$$

Substituting these values in equation (1), we have,

$$\sin(a+b) = \sin a \cos b + \cos a \sin b. \qquad \text{(A)}$$

This formula expresses the value of the sine of the sum of two arcs in terms of the sine and cosine of the single arcs. It is enunciated as follows:

The sine of the sum of two arcs or angles is equal to the sine of the first into the cosine of the second, plus the cosine of the first into the sine of the second.

2. If in formula (A) we substitute $-b$ for b, we have,

$$\sin(a-b) = \sin a \cos(-b) + \cos a \sin(-b);$$

but (Art. 80) $\cos(-b) = \cos b$, and $\sin(-b) = -\sin b$;

hence, $$\sin(a-b) = \sin a \cos b - \cos a \sin b. \qquad \text{(B)}$$

3. If in formula (B) we substitute $90°-a$ for a, we have,

$$\sin(90°-a-b)=\sin(90°-a)\cos b-\cos(90°-a)\sin b;$$

but, $\sin(90°-a-b)=\sin(90°-(a+b))=\cos(a+b)$,

and, $\sin(90°-a)=\cos a$, and $\cos(90-a)=\sin a$;

hence, $\cos(a+b)=\cos a\cos b-\sin a\sin b$. (C)

4. Substituting $-b$ for b in formula (C), we have,

$$\cos(a-b)=\cos a\cos(-b)-\sin a\sin(-b),$$

or, $\cos(a-b)=\cos a\cos b+\sin a\sin b$. (D)

5. From Table I., For. (6), and formulas (A) and (C), we have,

$$\tan(a+b)=\frac{\sin(a+b)}{\cos(a+b)}=\frac{\sin a\cos b+\cos a\sin b}{\cos a\cos b-\sin a\sin b}.$$

Dividing both terms of the last member by $\cos a\cos b$, we have,

$$\tan(a+b)=\frac{\dfrac{\sin a\cos b}{\cos a\cos b}+\dfrac{\cos a\sin b}{\cos a\cos b}}{1-\dfrac{\sin a\sin b}{\cos a\cos b}}.$$

Cancelling common factors, and reducing, we have,

$$\tan(a+b)=\frac{\tan a+\tan b}{1-\tan a\tan b}. \qquad \text{(E)}$$

6. Substituting $-b$ for b in formula (E), and reducing, we have,

$$\tan(a-b)=\frac{\tan a-\tan b}{1+\tan a\tan b}. \qquad \text{(F)}$$

7. Dividing formula (C) by (A), and reducing as in (5), we have,

$$\cot(a+b)=\frac{\cot a\cot b-1}{\cot b+\cot a}. \qquad \text{(G)}$$

8. Substituting $-b$ for b in formula (G), and reducing, we have,

$$\cot(a-b)=\frac{\cot a\cot b+1}{\cot b-\cot a}. \qquad \text{(H)}$$

85. Formulas for double and half arcs.

1. Making $a=b$ in formulas (A), (C), (E), and (G), we have,

$$\sin 2a=2\sin a\cos a; \qquad \text{(A}'\text{)}$$

$$\cos 2a=\cos^2 a-\sin^2 a, \qquad \text{(C}'\text{)}$$

$$\tan 2a = \frac{2\tan a}{1-\tan^2 a}, \qquad (E')$$

$$\cot 2a = \frac{\cot^2 a - 1}{2\cot a}. \qquad (G')$$

2. If now in (C′) we put $1-\sin^2 a$ for $\cos^2 a$, and then $1-\cos^2 a$ for $\sin^2 a$, we have,

$$\cos 2a = 1 - 2\sin^2 a, \qquad (1)$$

$$\cos 2a = 2\cos^2 a - 1, \qquad (2)$$

from which we have,

$$\sin a = \sqrt{\frac{1-\cos 2a}{2}}, \qquad (A'')$$

$$\cos a = \sqrt{\frac{1+\cos 2a}{2}}. \qquad (C'')$$

Dividing (A″) by (C″) and then (C″) by (A″), multiplying numerator and denominator by the denominator, and reducing,

$$\tan a = \frac{\sin 2a}{1+\cos 2a}, \qquad (E'')$$

$$\cot a = \frac{\sin 2a}{1-\cos 2a}. \qquad (G'')$$

3. Now, substituting $\frac{1}{2}a$ for a in (A″), (C″), (E″), and (G″),

$$\sin \tfrac{1}{2}a = \sqrt{\frac{1-\cos a}{2}}, \qquad (A_1)$$

$$\cos \tfrac{1}{2}a = \sqrt{\frac{1+\cos a}{2}}, \qquad (C_1)$$

$$\tan \tfrac{1}{2}a = \frac{\sin a}{1+\cos a}, \qquad (E_1)$$

$$\cot \tfrac{1}{2}a = \frac{\sin a}{1-\cos a}. \qquad (G_1)$$

Taking the reciprocals of (E_1) and (G_1), we have,

$$\cot \tfrac{1}{2}a = \frac{1+\cos a}{\sin a}, \qquad (E_{11})$$

$$\tan \tfrac{1}{2}a = \frac{1-\cos a}{\sin a}. \qquad (G_{11})$$

86. ADDITIONAL FORMULAS.

1. Adding and subtracting formulas (A) and (B), and doing the same with (C) and (D), we have,

$$\sin(a+b)+\sin(a-b)=2\sin a\cos b, \qquad (1)$$
$$\sin(a+b)-\sin(a-b)=2\cos a\sin b, \qquad (2)$$
$$\cos(a+b)+\cos(a-b)=2\cos a\cos b, \qquad (3)$$
$$\cos(a-b)-\cos(a+b)=2\sin a\sin b. \qquad (4)$$

2. Now, making

$$a+b=p \text{ and } a-b=q,$$

whence, $a=\frac{1}{2}(p+q)$ and $b=\frac{1}{2}(p-q)$;

and substituting these in the above, and we have,

$$\sin p+\sin q=2\sin\tfrac{1}{2}(p+q)\cos\tfrac{1}{2}(p-q), \qquad (K)$$
$$\sin p-\sin q=2\cos\tfrac{1}{2}(p+q)\sin\tfrac{1}{2}(p-q), \qquad (L)$$
$$\cos p+\cos q=2\cos\tfrac{1}{2}(p+q)\cos\tfrac{1}{2}(p-q), \qquad (M)$$
$$\cos q-\cos p=2\sin\tfrac{1}{2}(p+q)\sin\tfrac{1}{2}(p-q). \qquad (N)$$

3. Now, dividing (K) by (L),

$$\frac{\sin p+\sin q}{\sin p-\sin q}=\frac{\sin\frac{1}{2}(p+q)\cos\frac{1}{2}(p-q)}{\cos\frac{1}{2}(p+q)\sin\frac{1}{2}(p-q)}=\frac{\tan\frac{1}{2}(p+q)}{\tan\frac{1}{2}(p-q)}. \qquad (P)$$

In a similar manner, we obtain

$$\frac{\sin p+\sin q}{\cos p+\cos q}=\frac{2\sin\frac{1}{2}(p+q)\cos\frac{1}{2}(p-q)}{2\cos\frac{1}{2}(p+q)\cos\frac{1}{2}(p-q)}=\tan\tfrac{1}{2}(p+q), \qquad (Q)$$
$$\frac{\sin p-\sin q}{\cos p+\cos q}=\frac{2\sin\frac{1}{2}(p-q)\cos\frac{1}{2}(p+q)}{2\cos\frac{1}{2}(p+q)\cos\frac{1}{2}(p-q)}=\tan\tfrac{1}{2}(p-q), \qquad (R)$$
$$\frac{\sin p+\sin q}{\sin(p+q)}=\frac{2\sin\frac{1}{2}(p+q)\cos\frac{1}{2}(p-q)}{2\sin\frac{1}{2}(p+q)\cos\frac{1}{2}(p+q)}=\frac{\cos\frac{1}{2}(p-q)}{\cos\frac{1}{2}(p+q)}, \qquad (S)$$
$$\frac{\sin p-\sin q}{\sin(p+q)}=\frac{2\sin\frac{1}{2}(p-q)\cos\frac{1}{2}(p+q)}{2\sin\frac{1}{2}(p+q)\cos\frac{1}{2}(p+q)}=\frac{\sin\frac{1}{2}(p-q)}{\sin\frac{1}{2}(p+q)}, \qquad (T)$$
$$\frac{\sin(p-q)}{\sin p-\sin q}=\frac{2\sin\frac{1}{2}(p-q)\cos\frac{1}{2}(p-q)}{2\sin\frac{1}{2}(p-q)\cos\frac{1}{2}(p+q)}=\frac{\cos\frac{1}{2}(p-q)}{\cos\frac{1}{2}(p+q)}. \qquad (U)$$

These formulas may be enunciated in propositions; thus formula (P) gives,

The sum of the sines of two arcs is to the difference of their sines as the tangent of one-half of the sum of the arcs is to the tangent of one-half of their difference.

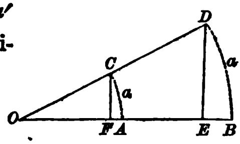
Comparing (S) and (U), we have,

$$\frac{\sin(p-q)}{\sin p-\sin q}=\frac{\sin p+\sin q}{\sin(p+q)}.$$

Hence, *the sine of the difference of two arcs is to the difference of their sines as the sum of the sines is to the sine of the sum.*

INTRODUCTION OF THE RADIUS.

87. In the preceding formulas, the radius, being unity, does not appear in any of the terms. When the radius is other than a unit, it should appear in these formulas, and we will now show how it may be introduced.



Let a be an arc whose radius is 1, and a' be an arc whose radius is R; then, by similar triangles,

$$\sin a : \sin a' :: 1 : R;$$

hence, $\sin a' = R \times \sin a$; $\sin a = \dfrac{\sin a'}{R}$;

and the same may be shown for the other circular functions.

Therefore, *any circular function whose radius is R is equal to the circular function whose radius is* 1, *multiplied by R.*

Also, *any circular function whose radius is* 1 *is equal to the circular function whose radius is R, divided by R.*

Now, if we substitute these in any of the formulas, we will find that R will be introduced in such a manner as to make the formulas *homogeneous*. Thus, For. 6, Tab. I., gives,

$$\frac{\tan a'}{R}=\frac{\dfrac{\sin a'}{R}}{\dfrac{\cos a'}{R}};\ \text{or, } \tan a'=\frac{R\sin a'}{\cos a'}.$$

Here, $\tan a'$ is a *line*, and $R \sin a' \div \cos a'$ is a *surface* divided by a *line*, which is also a *line*; hence, the formula is homogeneous. And since the same is generally true, therefore, we can introduce the radius in any formula by multiplying or dividing by R, so as to make the formula homogeneous.

CALCULATION OF A TABLE OF NATURAL SINES.

88. The circumference of a circle whose diameter is 1 is 3.14159 . . . . ; hence, when the *radius* is 1, the *semi-circumference* is 3.14159 . . . . . ; and if we divide this by 10800, the number of minutes in 180°, the quotient, .000290888 . . . . ., will be the length of an arc of *one minute.* Now, this arc is so small that it does not differ materially from its *sine;* hence, we may assume .000290888 . . . . . as the *sine of one minute.*

We then find the cosine of 1′ by For. 3, Table I. Thus,

$$\cos 1' = \sqrt{1 - \sin^2 1'} = .999999957 \ldots \ldots \qquad (1)$$

To find the sine of other arcs, we take the formula under Art. 86, putting it in the form,

$$\sin (a + b) = 2 \sin a \cos b - \sin (a - b).$$

Now, make $b = 1'$, and then in succession, a equal to 1′, 2′, 3′, etc., and we have,

$$\sin 2' = 2 \sin 1' \cos 1' - \sin 0 \ = .0005817764 \ldots \ldots$$
$$\sin 3' = 2 \sin 2' \cos 1' - \sin 1' = .0008726646 \ldots \ldots$$
$$\sin 4' = \text{etc.}$$

We may thus obtain the sines of any number of degrees and minutes up to 45°, the corresponding cosines being obtained from equation (1).

Then, since the sine of an arc equals the cosine of its complement, etc., the sines and cosines of arcs between 45° and 90° are immediately derived from those between 0° and 45°.

The tangents are found by dividing the sines by the cosines; the cotangents are found by dividing the cosines by the sines, or by dividing 1 by the tangents.

CALCULATION OF A TABLE OF LOGARITHMIC SINES.

89. A table of logarithmic sines is computed from a table of natural sines. The process is as follows:

For the logarithmic sine, *take the logarithm of the natural sine, and add* 10.

For, let $\sin a$ represent the natural sine, and let $\text{Sin } a$ represent the sine to a radius of 10,000,000,000; then, Art. 87,

$$\text{Sin } a = \sin a \times R;$$

taking logarithms, we have,

$$\log \text{Sin}\, a = \log \sin a + \log R.$$

But $\log R = \log 10{,}000{,}000{,}000 = 10$.

Hence, $\qquad \log \text{Sin}\, a = \log \sin a + 10.$

In the same manner, we find the log cosine; and in a similar manner, from the formulas of Table I., we can find all the other logarithmic circular functions.

THEOREMS AND PROBLEMS.

We now present a few exercises for original thought. The first and third are derived from a diagram; the 5th by For. 2. Art. 84; several which follow, by substituting values from Table I., obtaining an equation involving but one unknown quantity, which can then readily be found; the others, by judicious substitutions and reductions.

1. Prove that $\sin 60° = \frac{1}{2}\sqrt{3}$, and $\cos 60° = \frac{1}{2}$.
2. Prove that $\sin 30° = \frac{1}{2}$, and $\cos 30° = \frac{1}{2}\sqrt{3}$.
3. Prove that sin and cos of 45° equal $\frac{1}{2}\sqrt{2}$.
4. Prove that $\tan 45° = 1$, and $\sec 45° = \sqrt{2}$.
5. Prove $\sin 15°$, or $\sin(60°—45°) = \dfrac{\sqrt{3}-1}{2\sqrt{2}}$, and $\cos 15° = \dfrac{\sqrt{3}+1}{2\sqrt{2}}$.
6. Prove $\tan 15° = 2 - \sqrt{3}$, and $\cot 15° = 2 + \sqrt{3}$.
7. If $\sin a \cos a = \frac{1}{4}\sqrt{3}$ find $\sin a$ and $\cos a$.

   Ans. $\sin a = \frac{1}{2}\sqrt{3}$; $\cos a = \frac{1}{2}$.
8. If $3 \sin a + 5\sqrt{3} \times \cos a = 9$, find $\sin a$. *Ans.* $\sin a = \frac{1}{2}$ or $\frac{1}{7}$.
9. If $\sin a(\sin a - \cos a) = \frac{4}{25}$, find $\sin a$. *Ans.* $\sin a = \frac{4}{5}$.
10. If $\tan a = \frac{4}{3}$, find $\sin a$ and $\cos a$. *Ans.* $\sin a = \frac{4}{5}$; $\cos a = \frac{3}{5}$.
11. If $\tan a + \cot a = 2$, find $\tan a$. *Ans.* $\tan a = 1$.
12. Prove that $\tan^2 a - \sin^2 a = \tan^2 a \sin^2 a$.
13. Prove that $\sec^2 a \operatorname{cosec}^2 a = \sec^2 a + \operatorname{cosec}^2 a$.
14. Prove that $\sin(30° + a) + \sin(30° - a) = \cos a$.
15. Prove that $\cos(60° + a) + \cos(60° - a) = \cos a$.
16. If $a + b + c = 180°$, prove that

$$\tan a + \tan b + \tan c = \tan a \tan b \tan c.$$

17. If $a + b + c = 90°$, prove that

$$\cot a + \cot b + \cot c = \cot a \cot b \cot c.$$

SUGGESTION.—In 16th, $\tan(a + b) = \tan(180° - c)$, develop and simplify; and similarly in 17th.

A TABLE

OF

LOGARITHMS OF NUMBERS

FROM 1 TO 10,000.

N.	Log.	N.	Log.	N.	Log.	N.	Log.
1	0·000000	26	1·414973	51	1·707570	76	1·880814
2	0·301030	27	1·431364	52	1·716003	77	1·886491
3	0·477121	28	1·447158	53	1·724276	78	1·892085
4	0·602060	29	1·462398	54	1·732394	79	1·897627
5	0·698970	30	1·477121	55	1·740363	80	1·903090
6	0·778151	31	1·491362	56	1·748188	81	1·908485
7	0·845098	32	1·505150	57	1·755875	82	1·913814
8	0·903090	33	1·518514	58	1·763428	83	1·919078
9	0·954243	34	1·531479	59	1·770852	84	1·924279
10	1·000000	35	1·544068	60	1·778151	85	1·929419
11	1·041393	36	1·556303	61	1·785330	86	1·934498
12	1·079181	37	1·568202	62	1·792392	87	1·939519
13	1·113943	38	1·579784	63	1·799341	88	1·944483
14	1·146128	39	1·591065	64	1·806181	89	1·949390
15	1·176091	40	1·602060	65	1·812913	90	1·954243
16	1·204120	41	1·612784	66	1·819544	91	1·959041
17	1·230449	42	1·623249	67	1·826075	92	1·963788
18	1·255273	43	1·633468	68	1·832509	93	1·968483
19	1·278754	44	1·643453	69	1·838849	94	1·973128
20	1·301030	45	1·653213	70	1·845098	95	1·977724
21	1·322219	46	1·662758	71	1·851258	96	1·982271
22	1·342423	47	1·672098	72	1·857333	97	1·986772
23	1·361728	48	1·681241	73	1·863323	98	1·991226
24	1·380211	49	1·690196	74	1·869232	99	1·995635
25	1·397940	50	1·698970	75	1·875061	100	2·000000

REMARK.—In the following table, in the nine right-hand columns of each page, where the first or leading figures change from 9's to 0's, points or dots are introduced instead of the 0's, to catch the eye, and to indicate that from thence the two figures of the Logarithm to be taken from the second column, stand in the next line below.

N.	0	1	2	3	4	5	6	7	8	9	D.
100	000000	0434	0868	1301	1734	2166	2598	3029	3461	3891	432
101	4321	4751	5181	5609	6038	6466	6894	7321	7748	8174	428
102	8600	9026	9451	9876	•300	•724	1147	1570	1993	2415	424
103	012837	3259	3680	4100	4521	4940	5360	5779	6197	6616	419
104	7033	7451	7868	8284	8700	9116	9532	9947	•361	•775	416
105	021189	1603	2016	2428	2841	3252	3664	4075	4486	4896	412
106	5306	5715	6125	6533	6942	7350	7757	8164	8571	8978	408
107	9384	9789	•195	•600	1004	1408	1812	2216	2619	3021	404
108	033424	3826	4227	4628	5029	5430	5830	6230	6629	7028	400
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555	4293	4371	4449	4528	4606	4684	4762	4840	4919	4997	78
556	5075	5153	5231	5309	5387	5465	5543	5621	5699	5777	78
557	5855	5933	6011	6089	6167	6245	6323	6401	6479	6556	78
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782	3207	3262	3318	3373	3429	3484	3540	3595	3651	3706	56
783	3762	3817	3873	3928	3984	4039	4094	4150	4205	4261	55
784	4316	4371	4427	4482	4538	4593	4648	4704	4759	4814	55
785	4870	4925	4980	5036	5091	5146	5201	5257	5312	5367	55
786	5423	5478	5533	5588	5644	5699	5754	5809	5864	5920	55
787	5975	6030	6085	6140	6195	6251	6306	6361	6416	6471	55
788	6526	6581	6636	6692	6747	6802	6857	6912	6967	7022	55
789	7077	7132	7187	7242	7297	7352	7407	7462	7517	7572	55
790	897627	7682	7737	7792	7847	7902	7957	8012	8067	8122	55
791	8176	8231	8286	8341	8396	8451	8506	8561	8615	8670	55
792	8725	8780	8835	8890	8944	8999	9054	9109	9164	9218	55
793	9273	9328	9383	9437	9492	9547	9602	9656	9711	9766	55
794	9821	9875	9930	9985	••39	••94	•149	•203	•258	•312	55
795	900367	0422	0476	0531	0586	0640	0695	0749	0804	0859	55
796	0913	0968	1022	1077	1131	1186	1240	1295	1349	1404	55
797	1458	1513	1567	1622	1676	1731	1785	1840	1894	1948	54
798	2003	2057	2112	2166	2221	2275	2329	2384	2438	2492	54
799	2547	2601	2655	2710	2764	2818	2873	2927	2981	3036	54
800	903090	3144	3199	3253	3307	3361	3416	3470	3524	3578	54
801	3633	3687	3741	3795	3849	3904	3958	4012	4066	4120	54
802	4174	4229	4283	4337	4391	4445	4499	4553	4607	4661	54
803	4716	4770	4824	4878	4932	4986	5040	5094	5148	5202	54
804	5256	5310	5364	5418	5472	5526	5580	5634	5688	5742	54
805	5796	5850	5904	5958	6012	6066	6119	6173	6227	6281	54
806	6335	6389	6443	6497	6551	6604	6658	6712	6766	6820	54
807	6874	6927	6981	7035	7089	7143	7196	7250	7304	7358	54
808	7411	7465	7519	7573	7626	7680	7734	7787	7841	7895	54
809	7949	8002	8056	8110	8163	8217	8270	8324	8378	8431	54
810	908485	8539	8592	8646	8699	8753	8807	8860	8914	8967	54
811	9021	9074	9128	9181	9235	9289	9342	9396	9449	9503	54
812	9556	9610	9663	9716	9770	9823	9877	9930	9984	••37	53
813	910091	0144	0197	0251	0304	0358	0411	0464	0518	0571	53
814	0624	0678	0731	0784	0838	0891	0944	0998	1051	1104	53
815	1158	1211	1264	1317	1371	1424	1477	1530	1584	1637	53
816	1690	1743	1797	1850	1903	1956	2009	2063	2116	2169	53
817	2222	2275	2328	2381	2435	2488	2541	2594	2647	2700	53
818	2753	2806	2859	2913	2966	3019	3072	3125	3178	3231	53
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821	4343	4396	4449	4502	4555	4608	4660	4713	4766	4819	53
822	4872	4925	4977	5030	5083	5136	5189	5241	5294	5347	53
823	5400	5453	5505	5558	5611	5664	5716	5769	5822	5875	53
824	5927	5980	6033	6085	6138	6191	6243	6296	6349	6401	53
825	6454	6507	6559	6612	6664	6717	6770	6822	6875	6927	53
826	6980	7033	7085	7138	7190	7243	7295	7348	7400	7453	53
827	7506	7558	7611	7663	7716	7768	7820	7873	7925	7978	52
828	8030	8083	8135	8188	8240	8293	8345	8397	8450	8502	52
829	8555	8607	8659	8712	8764	8816	8869	8921	8973	9026	52
830	919078	9130	9183	9235	9287	9340	9392	9444	9496	9549	52
831	9601	9653	9706	9758	9810	9862	9914	9967	••19	••71	52
832	920123	0176	0228	0280	0332	0384	0436	0489	0541	0593	52
833	0645	0697	0749	0801	0853	0906	0958	1010	1062	1114	52
834	1166	1218	1270	1322	1374	1426	1478	1530	1582	1634	52
835	1686	1738	1790	1842	1894	1946	1998	2050	2102	2154	52
836	2206	2258	2310	2362	2414	2466	2518	2570	2622	2674	52
837	2725	2777	2829	2881	2933	2985	3037	3089	3140	3192	52
838	3244	3296	3348	3399	3451	3503	3555	3607	3658	3710	52
839	3762	3814	3865	3917	3969	4021	4072	4124	4176	4228	52
840	924279	4331	4383	4434	4486	4538	4589	4641	4693	4744	52
841	4796	4848	4899	4951	5003	5054	5106	5157	5209	5261	52
842	5312	5364	5415	5467	5518	5570	5621	5673	5725	5776	52
843	5828	5879	5931	5982	6034	6085	6137	6188	6240	6291	51
844	6342	6394	6445	6497	6548	6600	6651	6702	6754	6805	51
845	6857	6908	6959	7011	7062	7114	7165	7216	7268	7319	51
846	7370	7422	7473	7524	7576	7627	7678	7730	7781	7832	51
847	7883	7935	7986	8037	8088	8140	8191	8242	8293	8345	51
848	8396	8447	8498	8549	8601	8652	8703	8754	8805	8857	51
849	8908	8959	9010	9061	9112	9163	9215	9266	9317	9368	51
850	929419	9470	9521	9572	9623	9674	9725	9776	9827	9879	51
851	9930	9981	••32	••83	•134	•185	•236	•287	•338	•389	51
852	930440	0491	0542	0592	0643	0694	0745	0796	0847	0898	51
853	0949	1000	1051	1102	1153	1204	1254	1305	1356	1407	51
854	1458	1509	1560	1610	1661	1712	1763	1814	1865	1915	51
855	1966	2017	2068	2118	2169	2220	2271	2322	2372	2423	51
856	2474	2524	2575	2626	2677	2727	2778	2829	2879	2930	51
857	2981	3031	3082	3133	3183	3234	3285	3335	3386	3437	51
858	3487	3538	3589	3639	3690	3740	3791	3841	3892	3943	51
859	3993	4044	4094	4145	4195	4246	4296	4347	4397	4448	51
860	934498	4549	4599	4650	4700	4751	4801	4852	4902	4953	50
861	5003	5054	5104	5154	5205	5255	5306	5356	5406	5457	50
862	5507	5558	5608	5658	5709	5759	5809	5860	5910	5960	50
863	6011	6061	6111	6162	6212	6262	6313	6363	6413	6463	50
864	6514	6564	6614	6665	6715	6765	6815	6865	6916	6966	50
865	7016	7066	7117	7167	7217	7267	7317	7367	7418	7468	50
866	7518	7568	7618	7668	7718	7769	7819	7869	7919	7969	50
867	8019	8069	8119	8169	8219	8269	8320	8370	8420	8470	50
868	8520	8570	8620	8670	8720	8770	8820	8870	8920	8970	50
869	9020	9070	9120	9170	9220	9270	9320	9369	9419	9469	50
870	939519	9569	9619	9669	9719	9769	9819	9869	9918	9968	50
871	940018	0068	0118	0168	0218	0267	0317	0367	0417	0467	50
872	0516	0566	0616	0666	0716	0765	0815	0865	0915	0964	50
873	1014	1064	1114	1163	1213	1263	1313	1362	1412	1462	50
874	1511	1561	1611	1660	1710	1760	1809	1859	1909	1958	50
875	2008	2058	2107	2157	2207	2256	2306	2355	2405	2455	50
876	2504	2554	2603	2653	2702	2752	2801	2851	2901	2950	50
877	3000	3049	3099	3148	3198	3247	3297	3346	3396	3445	59
878	3495	3544	3593	3643	3692	3742	3791	3841	3890	3939	59
879	3989	4038	4088	4137	4186	4236	4285	4335	4384	4433	59
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880	944483	4532	4581	4631	4680	4729	4779	4828	4877	4927	49
881	4976	5025	5074	5124	5173	5222	5272	5321	5370	5419	49
882	5469	5518	5567	5616	5665	5715	5764	5813	5862	5912	49
883	5961	6010	6059	6108	6157	6207	6256	6305	6354	6403	49
884	6452	6501	6551	6600	6649	6698	6747	6796	6845	6894	49
885	6943	6992	7041	7090	7140	7189	7238	7287	7336	7385	49
886	7434	7483	7532	7581	7630	7679	7728	7777	7826	7875	49
887	7924	7973	8022	8070	8119	8168	8217	8266	8315	8364	49
888	8413	8462	8511	8560	8609	8657	8706	8755	8804	8853	49
889	8902	8951	8999	9048	9097	9146	9195	9244	9292	9341	49
890	949390	9439	9488	9536	9585	9634	9683	9731	9780	9829	49
891	9878	9926	9975	••24	••73	•121	•170	•219	•267	•316	49
892	950365	0414	0462	0511	0560	0608	0657	0706	0754	0803	49
893	0851	0900	0949	0997	1046	1095	1143	1192	1240	1289	49
894	1338	1386	1435	1483	1532	1580	1629	1677	1726	1775	49
895	1823	1872	1920	1969	2017	2066	2114	2163	2211	2260	48
896	2308	2356	2405	2453	2502	2550	2599	2647	2696	2744	48
897	2792	2841	2889	2938	2986	3034	3083	3131	3180	3228	48
898	3276	3325	3373	3421	3470	3518	3566	3615	3663	3711	48
899	3760	3808	3856	3905	3953	4001	4049	4098	4146	4194	48
900	954243	4291	4339	4387	4435	4484	4532	4580	4628	4677	48
901	4725	4773	4821	4869	4918	4966	5014	5062	5110	5158	48
902	5207	5255	5303	5351	5399	5447	5495	5543	5592	5640	48
903	5688	5736	5784	5832	5880	5928	5976	6024	6072	6120	48
904	6168	6216	6265	6313	6361	6409	6457	6505	6553	6601	48
905	6649	6697	6745	6793	6840	6888	6936	6984	7032	7080	48
906	7128	7176	7224	7272	7320	7368	7416	7464	7512	7559	48
907	7607	7655	7703	7751	7799	7847	7894	7942	7990	8038	48
908	8086	8134	8181	8229	8277	8325	8373	8421	8468	8516	48
909	8564	8612	8659	8707	8755	8803	8850	8898	8946	8994	48
910	959041	9089	9137	9185	9232	9280	9328	9375	9423	9471	48
911	9518	9566	9614	9661	9709	9757	9804	9852	9900	9947	48
912	9995	••42	••90	•138	•185	•233	•280	•328	•376	•423	48
913	960471	0518	0566	0613	0661	0709	0756	0804	0851	0899	48
914	0946	0994	1041	1089	1136	1184	1231	1279	1326	1374	47
915	1421	1469	1516	1563	1611	1658	1706	1753	1801	1848	47
916	1895	1943	1990	2038	2085	2132	2180	2227	2275	2322	47
917	2369	2417	2464	2511	2559	2606	2653	2701	2748	2795	47
918	2843	2890	2937	2985	3032	3079	3126	3174	3221	3268	47
919	3316	3363	3410	3457	3504	3552	3599	3646	3693	3741	47
920	963788	3835	3882	3929	3977	4024	4071	4118	4165	4212	47
921	4260	4307	4354	4401	4448	4495	4542	4590	4637	4684	47
922	4731	4778	4825	4872	4919	4966	5013	5061	5108	5155	47
923	5202	5249	5296	5343	5390	5437	5484	5531	5578	5625	47
924	5672	5719	5766	5813	5860	5907	5954	6001	6048	6095	47
925	6142	6189	6236	6283	6329	6376	6423	6470	6517	6564	47
926	6611	6658	6705	6752	6799	6845	6892	6939	6986	7033	47
927	7080	7127	7173	7220	7267	7314	7361	7408	7454	7501	47
928	7548	7595	7642	7688	7735	7782	7829	7875	7922	7969	47
929	8016	8062	8109	8156	8203	8249	8296	8343	8390	8436	47
930	968483	8530	8576	8623	8670	8716	8763	8810	8856	8903	47
931	8950	8996	9043	9090	9136	9183	9229	9276	9323	9369	47
932	9416	9463	9509	9556	9602	9649	9695	9742	9789	9835	47
933	9882	9928	9975	••21	••68	•114	•161	•207	•254	•300	47
934	970347	0393	0440	0486	0533	0579	0626	0672	0719	0765	46
935	0812	0858	0904	0951	0997	1044	1090	1137	1183	1229	46
936	1276	1322	1369	1415	1461	1508	1554	1601	1647	1693	46
937	1740	1786	1832	1879	1925	1971	2018	2064	2110	2157	46
938	2203	2249	2295	2342	2388	2434	2481	2527	2573	2619	46
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940	973128	3174	3220	3266	3313	3359	3405	3451	3497	3543	46
941	3590	3636	3682	3728	3774	3820	3866	3913	3959	4005	46
942	4051	4097	4143	4189	4235	4281	4327	4374	4420	4466	46
943	4512	4558	4604	4650	4696	4742	4788	4834	4880	4926	46
944	4972	5018	5064	5110	5156	5202	5248	5294	5340	5386	46
945	5432	5478	5524	5570	5616	5662	5707	5753	5799	5845	46
946	5891	5937	5983	6029	6075	6121	6167	6212	6258	6304	46
947	6350	6396	6442	6488	6533	6579	6625	6671	6717	6763	46
948	6808	6854	6900	6946	6992	7037	7083	7129	7175	7220	46
949	7266	7312	7358	7403	7449	7495	7541	7586	7632	7678	46
950	977724	7769	7815	7861	7906	7952	7998	8043	8089	8135	46
951	8181	8226	8272	8317	8363	8409	8454	8500	8546	8591	46
952	8637	8683	8728	8774	8819	8865	8911	8956	9002	9047	46
953	9093	9138	9184	9230	9275	9321	9366	9412	9457	9503	46
954	9548	9594	9639	9685	9730	9776	9821	9867	9912	9958	46
955	980003	0049	0094	0140	0185	0231	0276	0322	0367	0412	45
956	0458	0503	0549	0594	0640	0685	0730	0776	0821	0867	45
957	0912	0957	1003	1048	1093	1139	1184	1229	1275	1320	45
958	1366	1411	1456	1501	1547	1592	1637	1683	1728	1773	45
959	1819	1864	1909	1954	2000	2045	2090	2135	2181	2226	45
960	982271	2316	2362	2407	2452	2497	2543	2588	2633	2678	45
961	2723	2769	2814	2859	2904	2949	2994	3040	3085	3130	45
962	3175	3220	3265	3310	3356	3401	3446	3491	3536	3581	45
963	3626	3671	3716	3762	3807	3852	3897	3942	3987	4032	45
964	4077	4122	4167	4212	4257	4302	4347	4392	4437	4482	45
965	4527	4572	4617	4662	4707	4752	4797	4842	4887	4932	45
966	4977	5022	5067	5112	5157	5202	5247	5292	5337	5382	45
967	5426	5471	5516	5561	5606	5651	5696	5741	5786	5830	45
968	5875	5920	5965	6010	6055	6100	6144	6189	6234	6279	45
969	6324	6369	6413	6458	6503	6548	6593	6637	6682	6727	45
970	986772	6817	6861	6906	6951	6996	7040	7085	7130	7175	45
971	7219	7264	7309	7353	7398	7443	7488	7532	7577	7622	45
972	7666	7711	7756	7800	7845	7890	7934	7979	8024	8068	45
973	8113	8157	8202	8247	8291	8336	8381	8425	8470	8514	45
974	8559	8604	8648	8693	8737	8782	8826	8871	8916	8960	45
975	9005	9049	9094	9138	9183	9227	9272	9316	9361	9405	45
976	9450	9494	9539	9583	9628	9672	9717	9761	9806	9850	44
977	9895	9939	9983	••28	••72	•117	•161	•206	•250	•294	44
978	990339	0383	0428	0472	0516	0561	0605	0650	0694	0738	44
979	0783	0827	0871	0916	0960	1004	1049	1093	1137	1182	44
980	991226	1270	1315	1359	1403	1448	1492	1536	1580	1625	44
981	1669	1713	1758	1802	1846	1890	1935	1979	2023	2067	44
982	2111	2156	2200	2244	2288	2333	2377	2421	2465	2509	44
983	2554	2598	2642	2686	2730	2774	2819	2863	2907	2951	44
984	2995	3039	3083	3127	3172	3216	3260	3304	3348	3392	44
985	3436	3480	3524	3568	3613	3657	3701	3745	3789	3833	44
986	3877	3921	3965	4009	4053	4097	4141	4185	4229	4273	44
987	4317	4361	4405	4449	4493	4537	4581	4625	4669	4713	44
988	4757	4801	4845	4889	4933	4977	5021	5065	5108	5152	44
989	5196	5240	5284	5328	5372	5416	5460	5504	5547	5591	44
990	995635	5679	5723	5767	5811	5854	5898	5942	5986	6030	44
991	6074	6117	6161	6205	6249	6293	6337	6380	6424	6468	44
992	6512	6555	6599	6643	6687	6731	6774	6818	6862	6906	44
993	6949	6993	7037	7080	7124	7168	7212	7255	7299	7343	44
994	7386	7430	7474	7517	7561	7605	7648	7692	7736	7779	44
995	7823	7867	7910	7954	7998	8041	8085	8129	8172	8216	44
996	8259	8303	8347	8390	8434	8477	8521	8564	8608	8652	44
997	8695	8739	8782	8826	8869	8913	8956	9000	9043	9087	44
998	9131	9174	9218	9261	9305	9348	9392	9435	9479	9522	44
999	9565	9609	9652	9696	9739	9783	9826	9870	9913	9957	43
N.	0	1	2	3	4	5	6	7	8	9	D.

A TABLE

OF

LOGARITHMIC

SINES AND TANGENTS

FOR EVERY

DEGREE AND MINUTE

OF THE QUADRANT.

REMARK. The minutes in the left-hand column of each page, increasing downwards, belong to the degrees at the top; and those increasing upwards, in the right-hand column, belong to the degrees below.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	0·000000		10·000000		0·000000		Infinite.	60
1	6·463726	5017·17	000000	·00	6·463726	5017·17	13·536274	59
2	764756	2934·85	000000	·00	764756	2934·83	235244	58
3	940847	2082·31	000000	·00	940847	2082·31	059153	57
4	7·065786	1615·17	000000	·00	7·065786	1615·17	12·934214	56
5	162696	1319·68	000000	·00	162696	1319·69	837304	55
6	241877	1115·75	9·999999	·01	241878	1115·78	758122	54
7	308824	966·53	999999	·01	308825	996·53	691175	53
8	366816	852·54	999999	·01	366817	852·54	633183	52
9	417968	762·63	999999	·01	417970	762·63	582030	51
10	463725	689·88	999998	·01	463727	689·88	536273	50
11	7·505118	629·81	9·999998	·01	7·505120	629·81	12·494880	49
12	542906	579·36	999997	·01	542909	579·33	457091	48
13	577668	536·41	999997	·01	577672	536·42	422328	47
14	609853	499·38	999996	·01	609857	499·39	390143	46
15	639816	467·14	999996	·01	639820	467·15	360180	45
16	667845	438·81	999995	·01	667849	438·82	332151	44
17	694173	413·72	999995	·01	694179	413·73	305821	43
18	718997	391·35	999994	·01	719004	391·36	280997	42
19	742477	371·27	999993	·01	742484	371·28	257516	41
20	764754	353·15	999993	·01	764761	351·36	235239	40
21	7·785943	336·72	9·999992	·01	7·785951	336·73	12·214049	39
22	806146	321·75	999991	·01	806155	321·76	193845	38
23	825451	308·05	999990	·01	825460	308·06	174540	37
24	843934	295·47	999989	·02	843944	295·49	156056	36
25	861662	283·88	999988	·02	861674	283·90	138326	35
26	878695	273·17	999988	·02	878708	273·18	121292	34
27	895085	263·23	999987	·02	895099	263·25	104901	33
28	910879	253·99	999986	·02	910894	254·01	089106	32
29	926119	245·38	999985	·02	926134	245·40	073866	31
30	940842	237·33	999983	·02	940858	237·35	059142	30
31	7·955082	229·80	9·999982	·02	7·955100	229·81	12·044900	29
32	968870	222·73	999981	·02	968889	222·75	031111	28
33	982233	216·08	999980	·02	982253	216·10	017747	27
34	995198	209·81	999979	·02	995219	209·83	004781	26
35	8·007787	203·90	999977	·02	8·007809	203·92	11·992191	25
36	020021	198·31	999976	·02	020045	198·33	979955	24
37	031919	193·02	999975	·02	031945	193·05	968055	23
38	043501	188·01	999973	·02	043527	188·03	956473	22
39	054781	183·25	999972	·02	054809	183·27	945191	21
40	065776	178·72	999971	·02	065806	178·74	934194	20
41	8·076500	174·41	9·999969	·02	8·076531	174·44	11·923469	19
42	086965	170·31	999968	·02	086997	170·34	913003	18
43	097183	166·39	999966	·02	097217	166·42	902783	17
44	107167	162·65	999964	·03	107202	162·68	892797	16
45	116926	159·08	999963	·03	116963	159·10	883037	15
46	126471	155·66	999961	·03	126510	155·68	873490	14
47	135810	152·38	999959	·03	135851	152·41	864149	13
48	144953	149·24	999958	·03	144996	149·27	855004	12
49	153907	146·22	999956	·03	153952	146·27	846048	11
50	162681	143·33	999954	·03	162727	143 36	837273	10
51	8·171280	140·54	9·999952	·03	8·171328	140 57	11·828672	9
52	179713	137·86	999950	·03	179763	137·90	820237	8
53	187985	135·29	999948	·03	188036	135·32	811964	7
54	196102	132·80	999946	·03	196156	132·84	803844	6
55	204070	130·41	999944	·03	204126	130·44	795874	5
56	211895	128·10	999942	·04	211953	128·14	788047	4
57	219581	125·87	999940	·04	219641	125·90	780359	3
58	227134	123·72	999938	·04	227195	123·76	772805	2
59	234557	121·64	999936	·04	234621	121·68	765379	1
60	241855	119·63	999934	·04	241921	119·67	758079	0
	Cosine	D.	Sine	89°	Cotang.	D.	Tang.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	8·241855	119·63	9·999934	·04	8·241921	119·67	11·758079	60
1	249033	117·68	999932	·04	249102	117·72	750898	59
2	256094	115·80	999929	·04	256165	115·84	743835	58
3	263042	113·98	999927	·04	263115	114·02	736885	57
4	269881	112·21	999925	·04	269956	112·25	730044	56
5	276614	110·50	999922	·04	276691	110·54	723309	55
6	283243	108·83	999920	·04	283323	108·87	716677	54
7	289773	107·21	999918	·04	289856	107·26	710144	53
8	296207	105·65	999915	·04	296292	105·70	703708	52
9	302546	104·13	999913	·04	302634	104·18	697366	51
10	308794	102·66	999910	·04	308884	102·70	691116	50
11	8·314904	101·22	9·999907	·04	8·315046	101·26	11·684954	49
12	321027	99·82	999905	·04	321122	99·87	678878	48
13	327016	98·47	999902	·04	327114	98·51	672886	47
14	332924	97·14	999899	·05	333025	97·19	666975	46
15	338753	95·86	999897	·05	338856	95·90	661144	45
16	344504	94·60	999894	·05	344610	94·65	655390	44
17	350181	93·38	999891	·05	350289	93·43	649711	43
18	355783	92·19	999888	·05	355895	92·24	644105	42
19	361315	91·03	999885	·05	361430	91·08	638570	41
20	366777	89·90	999882	·05	366895	89·95	633105	40
21	8·372171	88·80	9·999879	·05	8·372292	88·85	11·627708	39
22	377499	87·72	999876	·05	377622	87·77	622378	38
23	382762	86·67	999873	·05	382889	86·72	617111	37
24	387962	85·64	999870	·05	388092	85·70	611908	36
25	393101	84·64	999867	·05	393234	84·70	606766	35
26	398179	83·66	999864	·05	398315	83·71	601685	34
27	403199	82·71	999861	·05	403338	82·76	596662	33
28	408161	81·77	999858	·05	408304	81·82	591696	32
29	413068	80·86	999854	·05	413213	80·91	586787	31
30	417919	79·96	999851	·06	418068	80·02	581932	30
31	8·422717	79·09	9·999848	·06	8·422869	79·14	11·577131	29
32	427462	78·23	999844	·06	427618	78·30	572382	28
33	432156	77·40	999841	·06	432315	77·45	567685	27
34	436800	76·57	999838	·06	436962	76·63	563038	26
35	441394	75·77	999834	·06	441560	75·83	558440	25
36	445941	74·99	999831	·06	446110	75·05	553890	24
37	450440	74·22	999827	·06	450613	74·28	549387	23
38	454893	73·46	999823	·06	455070	73·52	544930	22
39	459301	72·73	999820	·06	459481	72·79	540519	21
40	463665	72·00	999816	·06	463849	72·06	536151	20
41	8·467985	71·29	9·999812	·06	8·468172	71·35	11·531828	19
42	472263	70·60	999809	·06	472454	70·66	527546	18
43	476498	69·91	999805	·06	476693	69·98	523307	17
44	480693	69·24	999801	·06	480892	69·31	519108	16
45	484848	68·59	999797	·07	485050	68·65	514950	15
46	488963	67·94	999793	·07	489170	68·01	510830	14
47	493040	67·31	999790	·07	493250	67·38	506750	13
48	497078	66·69	999786	·07	497293	66·76	502707	12
49	501080	66·08	999782	·07	501298	66·15	498702	11
50	505045	65·48	999778	·07	505267	65·55	494733	10
51	8·508974	64·89	9·999774	·07	8·509200	64·96	11·490800	9
52	512867	64·31	999769	·07	513098	64·39	486902	8
53	516726	63·75	999765	·07	516961	63·82	483039	7
54	520551	63·19	999761	·07	520790	63·26	479210	6
55	524343	62·64	999757	·07	524586	62·72	475414	5
56	528102	62·11	999753	·07	528349	62·18	471651	4
57	531828	61·58	999748	·07	532080	61·65	467920	3
58	535523	61·06	999744	·07	535779	61·13	464221	2
59	539186	60·55	999740	·07	539447	60·62	460553	1
60	542819	60·04	999735	·07	543084	60·12	456916	0
	Cosine	D.	Sine	88°	Cotang.	D.	Tang.	

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	8·542819	60·04	9·999735	·07	8·543084	60·12	11·456916	60
1	546422	59·55	999731	·07	546691	59·62	453309	59
2	549995	59·06	999726	·07	550268	59·14	449732	58
3	553539	58·58	999722	·08	553817	58·66	446183	57
4	557054	58·11	999717	·08	557336	58·19	442664	56
5	560540	57·65	999713	·08	560828	57·73	439172	55
6	563999	57·19	999708	·08	564291	57·27	435709	54
7	567431	56·74	999704	·08	567727	56·82	432273	53
8	570836	56·30	999699	·08	571137	56·38	428863	52
9	574214	55·87	999694	·08	574520	55·95	425480	51
10	577566	55·44	999689	·08	577877	55·52	422123	50
11	8·580892	55·02	9·999685	·08	8·581208	55·10	11·418792	49
12	584193	54·60	999680	·08	584514	54·68	415486	48
13	587469	54·19	999675	·08	587795	54·27	412205	47
14	590721	53·79	999670	·08	591051	53·87	408949	46
15	593948	53·39	999665	·08	594283	53·47	405717	45
16	597152	53·00	999660	·08	597492	53·08	402508	44
17	600332	52·61	999655	·08	600677	52·70	399323	43
18	603489	52·23	999650	·08	603839	52·32	396161	42
19	606623	51·86	999645	·09	606978	51·94	393022	41
20	609734	51·49	999640	·09	610094	51·58	389906	40
21	8·612823	51·12	9·999635	·09	8·613189	51·21	11·386811	39
22	615891	50·76	999629	·09	616262	50·85	383738	38
23	618937	50·41	999624	·09	619313	50·50	380687	37
24	621962	50·06	999619	·09	622343	50·15	377657	36
25	624965	49·72	999614	·09	625352	49·81	374648	35
26	627948	49·38	999608	·09	628340	49·47	371660	34
27	630911	49·04	999603	·09	631308	49·13	368692	33
28	633854	48·71	999597	·09	634256	48·80	365744	32
29	636776	48·39	999592	·09	637184	48·48	362816	31
30	639680	48·06	999586	·09	640093	48·16	359907	30
31	8·642563	47·75	9·999581	·09	8·642982	47·84	11·357018	29
32	645428	47·43	999575	·09	645853	47·53	354147	28
33	648274	47·12	999570	·09	648704	47·22	351296	27
34	651102	46·82	999564	·09	651537	46·91	348463	26
35	653911	46·52	999558	·10	654352	46·61	345648	25
36	656702	46·22	999553	·10	657149	46·31	342851	24
37	659475	45·92	999547	·10	659928	46·02	340072	23
38	662230	45·63	999541	·10	662689	45·73	337311	22
39	664968	45·35	999535	·10	665433	45·44	334567	21
40	667689	45·06	999529	·10	668160	45·26	331840	20
41	8·670393	44·79	9·999524	·10	8·670870	44·88	11·329130	19
42	673080	44·51	999518	·10	673563	44·61	326437	18
43	675751	44·24	999512	·10	676239	44·34	323761	17
44	678405	43·97	999506	·10	678900	44·17	321100	16
45	681043	43·70	999500	·10	681544	43·80	318456	15
46	683665	43·44	999493	·10	684172	43·54	315828	14
47	686272	43·18	999487	·10	686784	43·28	313216	13
48	688863	42·92	999481	·10	689381	43·03	310619	12
49	691438	42·67	999475	·10	691963	42·77	308037	11
50	693998	42·42	999469	·10	694529	42·52	305471	10
51	8·696543	42·17	9·999463	·11	8·697081	42·28	11·302919	9
52	699073	41·92	999456	·11	699617	42·03	300383	8
53	701589	41·68	999450	·11	702139	41·79	297861	7
54	704090	41·44	999443	·11	704646	41·55	295354	6
55	706577	41·21	999437	·11	707140	41·32	292860	5
56	709049	40·97	999431	·11	709618	41·08	290382	4
57	711507	40·74	999424	·11	712083	40·85	287917	3
58	713952	40·51	999418	·11	714534	40·62	285465	2
59	716383	40·29	999411	·11	716972	40·40	283028	1
60	718800	40·06	999404	·11	719396	40·17	280604	0
	Cosine	D.	Sine	87°	Cotang.	D.	Tang.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	8·718800	40·06	9·999404	·11	8·719396	40·17	11·280604	60
1	721204	39·84	999398	·11	721806	39·95	278194	59
2	723595	39·62	999391	·11	724204	39·74	275796	58
3	725972	39·41	999384	·11	726588	39·52	273412	57
4	728337	39·19	999378	·11	728959	39·30	271041	56
5	730688	38·98	999371	·11	731317	39·09	268683	55
6	733027	38·77	999364	·12	733663	38·89	266337	54
7	735354	38·57	999357	·12	735996	38·68	264004	53
8	737667	38·36	999350	·12	738317	38·48	261683	52
9	739969	38·16	999343	·12	740626	38·27	259374	51
10	742259	37·96	999336	·12	742922	38·07	257078	50
11	8·744536	37·76	9·999329	·12	8·745207	37·87	11·254793	49
12	746802	37·56	999322	·12	747479	37·68	252521	48
13	749055	37·37	999315	·12	749740	37·49	250260	47
14	751297	37·17	999308	·12	751989	37·29	248011	46
15	753528	36·98	999301	·12	754227	37·10	245773	45
16	755747	36·79	999294	·12	756453	36·92	243547	44
17	757955	36·61	999286	·12	758668	36·73	241332	43
18	760151	36·42	999279	·12	760872	36·55	239128	42
19	762337	36·24	999272	·12	763065	36·36	236935	41
20	764511	36·06	999265	·12	765246	36·18	234754	40
21	8·766675	35·88	9·999257	·12	8·767417	36·00	11·232583	39
22	768828	35·70	999250	·13	769578	35·83	230422	38
23	770970	35·53	999242	·13	771727	35·65	228273	37
24	773101	35·35	999235	·13	773866	35·48	226134	36
25	775223	35·18	999227	·13	775995	35·31	224005	35
26	777333	35·01	999220	·13	778114	35·14	221886	34
27	779434	34·84	999212	·13	780222	34·97	219778	33
28	781524	34·67	999205	·13	782320	34·80	217680	32
29	783605	34·51	999197	·13	784408	34·64	215592	31
30	785675	34·31	999189	·13	786486	34·47	213514	30
31	8·787736	34·18	9·999181	·13	8·788554	34·31	11·211446	29
32	789787	34·02	999174	·13	790613	34·15	209387	28
33	791828	33·86	999166	·13	792662	33·99	207338	27
34	793859	33·70	999158	·13	794701	33·83	205299	26
35	795881	33·54	999150	·13	796731	33·68	203269	25
36	797894	33·39	999142	·13	798752	33·52	201248	24
37	799897	33·23	999134	·13	800763	33·37	199237	23
38	801892	33·08	999126	·13	802765	33·22	197235	22
39	803876	32·93	999118	·13	804758	33·07	195242	21
40	805852	32·78	999110	·13	806742	32·92	193258	20
41	8·807819	32·63	9·999102	·13	8·808717	32·78	11·191283	19
42	809777	32·49	999094	·14	810683	32·62	189317	18
43	811726	32·34	999086	·14	812641	32·48	187359	17
44	813667	32·19	999077	·14	814589	32·33	185411	16
45	815599	32·05	999069	·14	816529	32·19	183471	15
46	817522	31·91	999061	·14	818461	32·05	181539	14
47	819436	31·77	999053	·14	820384	31·91	179616	13
48	821343	31·63	999044	·14	822298	31·77	177702	12
49	823240	31·49	999036	·14	824205	31·63	175795	11
50	825130	31·35	999027	·14	826103	31·50	173897	10
51	8·827011	31·22	9·999019	·14	8·827992	31·36	11·172008	9
52	828884	31·08	999010	·14	829874	31·23	170126	8
53	830749	30·95	999002	·14	831748	31·10	168252	7
54	832607	30·82	998993	·14	833613	30·96	166387	6
55	834456	30·69	998984	·14	835471	30·83	164529	5
56	836297	30·56	998976	·14	837321	30·70	162679	4
57	838130	30·43	998967	·15	839163	30·57	160837	3
58	839956	30·30	998958	·15	840998	30·45	159002	2
59	841774	30·17	998950	·15	842825	30·32	157175	1
60	843585	30·00	998941	·15	844644	30·19	155356	0
	Cosine	D	Sine	86°	Cotang.	D.	Tang.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	8·843585	30·05	9·998941	·15	8·844644	30·19	11·155356	60
1	845387	29·92	998932	·15	846455	30·07	153545	59
2	847183	29·80	998923	·15	848260	29·95	151740	58
3	848971	29·67	998914	·15	850057	29·82	149943	57
4	850751	29·55	998905	·15	851846	29·70	148154	56
5	852525	29·43	998896	·15	853628	29·58	146372	55
6	854291	29·31	998887	·15	855403	29·46	144597	54
7	856049	29·19	998878	·15	857171	29·35	142829	53
8	857801	29·07	998869	·15	858932	29·23	141068	52
9	859546	28·96	998860	·15	860686	29·11	139314	51
10	861283	28·84	998851	·15	862433	29·00	137567	50
11	8·863014	28·73	9·998841	·15	8·864173	28·88	11·135827	49
12	864738	28·61	998832	·15	865906	28·77	134094	48
13	866455	28·50	998823	·16	867632	28·66	132368	47
14	868165	28·39	998813	·16	869351	28·54	130649	46
15	869868	28·28	998804	·16	871064	28·43	128936	45
16	871565	28·17	998795	·16	872770	28·32	127230	44
17	873255	28·06	998785	·16	874469	28·21	125531	43
18	874938	27·95	998776	·16	876162	28·11	123838	42
19	876615	27·86	998766	·16	877849	28·00	122151	41
20	878285	27·73	998757	·16	879529	27·89	120471	40
21	8·879949	27·63	9·998747	·16	8·881202	27·79	11 118798	39
22	881607	27·52	998738	·16	882869	27·68	117131	38
23	883258	27·42	998728	·16	884530	27·58	115470	37
24	884903	27·31	998718	·16	886185	27·47	113815	36
25	886542	27·21	998708	·16	887833	27·37	112167	35
26	888174	27·11	998699	·16	889476	27·27	110524	34
27	889801	27·00	998689	·16	891112	27·17	108888	33
28	891421	26·90	998679	·16	892742	27·07	107258	32
29	893035	26·80	998669	·17	894366	26·97	105634	31
30	894643	26·70	998659	·17	895984	26·87	104016	30
31	8·896246	26·60	9·998649	·17	8·897596	26·77	11·102404	29
32	897842	26·51	998639	·17	899203	26·67	100797	28
33	899432	26·41	998629	·17	900803	26·58	099197	27
34	901017	26·31	998619	·17	902398	26·48	097602	26
35	902596	26·22	998609	·17	903987	26·38	096013	25
36	904169	26·12	998599	·17	905570	26·29	094430	24
37	905736	26·03	998589	·17	907147	26·20	092853	23
38	907297	25·93	998578	·17	908719	26·10	091281	22
39	908853	25·84	998568	·17	910285	26·01	089715	21
40	910404	25·75	998558	·17	911846	25·92	088154	20
41	8·911949	25·66	9·998548	·17	8·913401	25·83	11·086599	19
42	913488	25·56	998537	·17	914951	25·74	085049	18
43	915022	25·47	998527	·17	916495	25·65	083505	17
44	916550	25·38	998516	·18	918034	25·56	081966	16
45	918073	25·29	998506	·18	919568	25·47	080432	15
46	919591	25·20	998495	·18	921096	25·38	078904	14
47	921103	25·12	998485	·18	922619	25·30	077381	13
48	922610	25·03	998474	·18	924136	25·21	075864	12
49	924112	24·94	998464	·18	925649	25·12	074351	11
50	925609	24·86	998453	·18	927156	25·03	072844	10
51	8·927100	24·77	9·998442	·18	8·928658	24·95	11·071342	9
52	928587	24·69	998431	·18	930155	24·86	069845	8
53	930068	24·60	998421	·18	931647	24·78	068353	7
54	931544	24·52	998410	·18	933134	24·70	066866	6
55	933015	24·43	998399	·18	934616	24·61	065384	5
56	934481	24·35	998388	·18	936093	24·53	063907	4
57	935942	24·27	998377	·18	937565	24·45	062435	3
58	937398	24·19	998366	·18	939032	24·37	060968	2
59	938850	24·11	998355	·18	940494	24·30	059506	1
60	940296	24·03	998344	·18	941952	24·21	058048	0
	Cosine	D.	Sine	85°	Cotang.	D.	Tang.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	8·940296	24·03	9·998344	·19	8·941952	24·21	11·058048	60
1	941738	23·94	998333	·19	943404	24·13	056596	59
2	943174	23·87	998322	·19	944852	24·05	055148	58
3	944606	23·79	998311	·19	946295	23·97	053705	57
4	946034	23·71	998300	·19	947734	23·90	052266	56
5	947456	23·63	998289	·19	949168	23·82	050832	55
6	948874	23·55	998277	·19	950597	23·74	049403	54
7	950287	23·48	998266	·19	952021	23·66	047979	53
8	951696	23·40	998255	·19	953441	23·60	046559	52
9	953100	23·32	998243	·19	954856	23·51	045144	51
10	954499	23·25	998232	·19	956267	23·44	043733	50
11	8·955894	23·17	9·998220	·19	8·957674	23·37	11·042326	49
12	957284	23·10	998209	·19	959075	23·29	040925	48
13	958670	23·02	998197	·19	960473	23·23	039527	47
14	960052	22·95	998186	·19	961866	23·14	038134	46
15	961429	22·88	998174	·19	963255	23·07	036745	45
16	962801	22·80	998163	·19	964639	23·00	035361	44
17	964170	22·73	998151	·19	966019	22·93	033981	43
18	965534	22·66	998139	·20	967394	22·86	032606	42
19	966893	22·59	998128	·20	968766	22·79	031234	41
20	968249	22·52	998116	·20	970133	22·71	029867	40
21	8·969600	22·44	9·998104	·20	8·971496	22·65	11·028504	39
22	970947	22·38	998092	·20	972855	22·57	027145	38
23	972289	22·31	998080	·20	974209	22·51	025791	37
24	973628	22·24	998068	·20	975560	22·44	024440	36
25	974962	22·17	998056	·20	976906	22·37	023094	35
26	976293	22·10	998044	·20	978248	22·30	021752	34
27	977619	22·03	998032	·20	979586	22·23	020414	33
28	978941	21·97	998020	·20	980921	22·17	019079	32
29	980259	21·90	998008	·20	982251	22·10	017749	31
30	981573	21·83	997996	·20	983577	22·04	016423	30
31	8·982883	21·77	9·997985	·20	8·984899	21·97	11·015101	29
32	984189	21·70	997972	·20	986217	21·91	013783	28
33	985491	21·63	997959	·20	987532	21·84	012468	27
34	986789	21·57	997947	·20	988842	21·78	011158	26
35	988083	21·50	997935	·21	990149	21·71	009851	25
36	989374	21·44	997922	·21	991451	21·65	008549	24
37	990660	21·38	997910	·21	992750	21·58	007250	23
38	991943	21·31	997897	·21	994045	21·52	005955	22
39	993222	21·25	997885	·21	995337	21·46	004663	21
40	994497	21·19	997872	·21	996624	21·40	003376	20
41	8·995768	21·12	9·997860	·21	8·997908	21·34	11·002092	19
42	997036	21·06	997847	·21	999188	21·27	000812	18
43	998299	21·00	997835	·21	9·000465	21·21	10·999535	17
44	999560	20·94	997822	·21	001738	21·15	998262	16
45	9·000816	20·87	997809	·21	003007	21·09	996993	15
46	002069	20·82	997797	·21	004272	21·03	995728	14
47	003318	20·76	997784	·21	005534	20·97	994466	13
48	004563	20·70	997771	·21	006792	20·91	993208	12
49	005805	20·64	997758	·21	008047	20·85	991953	11
50	007044	20·58	997745	·21	009298	20·80	990702	10
51	9·008278	20·52	9·997732	·21	9·010546	20·74	10·989454	9
52	009510	20·46	997719	·21	011790	20·68	988210	8
53	010737	20·40	997706	·21	013031	20·62	986969	7
54	011962	20·34	997693	·22	014268	20·56	985732	6
55	013182	20·29	997680	·22	015502	20·51	984498	5
56	014400	20·23	997667	·22	016732	20·45	983268	4
57	015613	20·17	997654	·22	017959	20·40	982041	3
58	016824	20·12	997641	·22	019183	20·33	980817	2
59	018031	20·06	997628	·22	020403	20·28	979597	1
60	019235	20·00	997614	·22	021620	20·23	978380	0
	Cosine	D.	Sine	84°	Cotang.	D.	Tang.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9·019235	20·00	9·997614	·22	9·021620	20·23	10·978380	60
1	020435	19·95	997601	·22	022834	20·17	977166	59
2	021632	19·89	997588	·22	024044	20·11	975956	58
3	022825	19·84	997574	·22	025251	20·06	974749	57
4	024016	19·78	997561	·22	026455	20·00	973545	56
5	025203	19·73	997547	·22	027655	19·95	972345	55
6	026386	19·67	997534	·23	028852	19·90	971148	54
7	027567	19·62	997520	·23	030046	19·85	969954	53
8	028744	19·57	997507	·23	031237	19·79	968763	52
9	029918	19·51	997493	·23	032425	19·74	967575	51
10	031089	19·47	997480	·23	033609	19·69	966391	50
11	9·032257	19·41	9·997466	·23	9·034791	19·64	10·965209	49
12	033421	19·36	997452	·23	035969	19·58	964031	48
13	034582	19·30	997439	·23	037144	19·53	962856	47
14	035741	19·25	997425	·23	038316	19·48	961684	46
15	036896	19·20	997411	·23	039485	19·43	960515	45
16	038048	19·15	997397	·23	040651	19·38	959349	44
17	039197	19·10	997383	·23	041813	19·33	958187	43
18	040342	19·05	997369	·23	042973	19·28	957027	42
19	041485	18·99	997355	·23	044130	19·23	955870	41
20	042625	18·94	997341	·23	045284	19·18	954716	40
21	9·043762	18·89	9·997327	·24	9·046434	19·13	10·953566	39
22	044895	18·84	997313	·24	047582	19·08	952418	38
23	046026	18·79	997299	·24	048727	19·03	951273	37
24	047154	18·75	997285	·24	049869	18·98	950131	36
25	048279	18·70	997271	·24	051008	18·93	948992	35
26	049400	18·65	997257	·24	052144	18·89	947856	34
27	050519	18·60	997242	·24	053277	18·84	946723	33
28	051635	18·55	997228	·24	054407	18·79	945593	32
29	052749	18·50	997214	·24	055535	18·74	944465	31
30	053859	18·45	997199	·24	056659	18·70	943341	30
31	9·054966	18·41	9·997185	·24	9·057781	18·65	10·942219	29
32	056071	18·36	997170	·24	058900	18·60	941100	28
33	057172	18·31	997156	·24	060016	18·55	939984	27
34	058271	18·27	997141	·24	061130	18·51	938870	26
35	059367	18·22	997127	·24	062240	18·46	937760	25
36	060460	18·17	997112	·24	063348	18·42	936652	24
37	061551	18·13	997098	·24	064453	18·37	935547	23
38	062639	18·08	997083	·25	065556	18·33	934444	22
39	063724	18·04	997068	·25	066655	18·28	933345	21
40	064806	17·99	997053	·25	067752	18·24	932248	20
41	9·065885	17·94	9·997039	·25	9·068846	18·19	10·931154	19
42	066962	17·90	997024	·25	069938	18·15	930062	18
43	068036	17·86	997009	·25	071027	18·10	928973	17
44	069107	17·81	996994	·25	072113	18·06	927887	16
45	070176	17·77	996979	·25	073197	18·02	926803	15
46	071242	17·72	996964	·25	074278	17·97	925722	14
47	072306	17·68	996949	·25	075356	17·93	924644	13
48	073366	17·63	996934	·25	076432	17·89	923568	12
49	074424	17·59	996919	·25	077505	17·84	922495	11
50	075480	17·55	996904	·25	078576	17·80	921424	10
51	9·076533	17·50	9·996889	·25	9·079644	17·76	10·920356	9
52	077583	17·46	996874	·25	080710	17·72	919290	8
53	078631	17·42	996858	·25	081773	17·67	918227	7
54	079676	17·38	996843	·25	082833	17·63	917167	6
55	080719	17·33	996828	·25	083891	17·59	916109	5
56	081759	17·29	996812	·26	084947	17·55	915053	4
57	082797	17·25	996797	·26	086000	17·51	914000	3
58	083832	17·21	996782	·26	087050	17·47	912950	2
59	084864	17·17	996766	·26	088098	17·43	911902	1
60	085894	17·13	996751	·26	089144	17·38	910856	0
	Cosine	D.	Sine	83°	Cotang.	D.	Tang.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9·085894	17·13	9·996751	·26	9·089144	17·38	10·910856	60
1	086922	17·09	996735	·26	090187	17·34	909813	59
2	087947	17·04	996720	·26	091228	17·30	908772	58
3	088970	17·00	996704	·26	092266	17·27	907734	57
4	089990	16·96	996688	·26	093302	17·22	906698	56
5	091008	16·92	996673	·26	094336	17·19	905664	55
6	092024	16·88	996657	·26	095367	17·15	904633	54
7	093037	16·84	996641	·26	096395	17·11	903605	53
8	094047	16·80	996625	·26	097422	17·07	902578	52
9	095056	16·76	996610	·26	098446	17·03	901554	51
10	096062	16·73	996594	·26	099468	16·99	900532	50
11	9·097065	16·68	9·996578	·27	9·100487	16·95	10·899513	49
12	098066	16·65	996562	·27	101504	16·91	898496	48
13	099065	16·61	996546	·27	102519	16·87	897481	47
14	100062	16·57	996530	·27	103532	16·84	896468	46
15	101056	16·53	996514	·27	104542	16·80	895458	45
16	102048	16·49	996498	·27	105550	16·76	894450	44
17	103037	16·45	996482	·27	106556	16·72	893444	43
18	104025	16·41	996465	·27	107559	16·69	892441	42
19	105010	16·38	996449	·27	108560	16·65	891440	41
20	105992	16·34	996433	·27	109559	16·61	890441	40
21	9·106973	16·30	9·996417	·27	9·110556	16·58	10·889444	39
22	107951	16·27	996400	·27	111551	16·54	888449	38
23	108927	16·23	996384	·27	112543	16·50	887457	37
24	109901	16·19	996368	·27	113533	16·46	886467	36
25	110873	16·16	996351	·27	114521	16·43	885479	35
26	111842	16·12	996335	·27	115507	16·39	884493	34
27	112809	16·08	996318	·27	116491	16·36	883509	33
28	113774	16·05	996302	·28	117472	16·32	882528	32
29	114737	16·01	996285	·28	118452	16·29	881548	31
30	115698	15·97	996269	·28	119429	16·25	880571	30
31	9·116656	15·94	9·996252	·28	9·120404	16·22	10·879596	29
32	117613	15·90	996235	·28	121377	16·18	878623	28
33	118567	15·87	996219	·28	122348	16·15	877652	27
34	119519	15·83	996202	·28	123317	16·11	876683	26
35	120469	15·80	996185	·28	124284	16·07	875716	25
36	121417	15·76	996168	·28	125249	16·04	874751	24
37	122362	15·73	996151	·28	126211	16·01	873789	23
38	123306	15·69	996134	·28	127172	15·97	872828	22
39	124248	15·66	996117	·28	128130	15·94	871870	21
40	125187	15·62	996100	·28	129087	15·91	870913	20
41	9·126125	15·59	9·996083	·29	9·130041	15·87	10·869959	19
42	127060	15·56	996066	·29	130994	15·84	869006	18
43	127993	15·52	996049	·29	131944	15·81	868056	17
44	128925	15·49	996032	·29	132893	15·77	867107	16
45	129854	15·45	996015	·29	133839	15·74	866161	15
46	130781	15·42	995998	·29	134784	15·71	865216	14
47	131706	15·39	995980	·29	135726	15·67	864274	13
48	132630	15·35	995963	·29	136667	15·64	863333	12
49	133551	15·32	995946	·29	137605	15·61	862395	11
50	134470	15·29	995928	·29	138542	15·58	861458	10
51	9·135387	15·25	9·995911	·29	9·139476	15·55	10·860524	9
52	136303	15·22	995894	·29	140409	15·51	859591	8
53	137216	15·19	995876	·29	141340	15·48	858660	7
54	138128	15·16	995859	·29	142269	15·45	857731	6
55	139037	15·12	995841	·29	143196	15·42	856804	5
56	139944	15·09	995823	·29	144121	15·39	855879	4
57	140850	15·06	995806	·29	145044	15·35	854956	3
58	141754	15·03	995788	·29	145966	15·32	854034	2
59	142655	15·00	995771	·29	146885	15·29	853115	1
60	143555	14·96	995753	·29	147803	15·26	852197	0
	Cosine	D.	Sine	82°	Cotang.	D.	Tang.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9·143555	14·96	9·995753	·30	9·147803	15·26	10·852197	60
1	144453	14·93	995735	·30	148718	15·23	851282	59
2	145349	14·90	995717	·30	149632	15·20	850368	58
3	146243	14·87	995699	·30	150544	15·17	849456	57
4	147136	14·84	995681	·30	151454	15·14	848546	56
5	148026	14·81	995664	·30	152363	15·11	847637	55
6	148915	14·78	995646	·30	153269	15·08	846731	54
7	149802	14·75	995628	·30	154174	15·05	845826	53
8	150686	14·72	995610	·30	155077	15·02	844923	52
9	151569	14·69	995591	·30	155978	14·99	844022	51
10	152451	14·66	995573	·30	156877	14·96	843123	50
11	9·153330	14·63	9·995555	·30	9·157775	14·93	10·842225	49
12	154208	14·60	995537	·30	158671	14·90	841329	48
13	155083	14·57	995519	·30	159565	14·87	840435	47
14	155957	14·54	995501	·31	160457	14·84	839543	46
15	156830	14·51	995482	·31	161347	14·81	838653	45
16	157700	14·48	995464	·31	162236	14·79	837764	44
17	158569	14·45	995446	·31	163123	14·76	836877	43
18	159435	14·42	995427	·31	164008	14·73	835992	42
19	160301	14·39	995409	·31	164892	14·70	835108	41
20	161164	14·36	995390	·31	165774	14·67	834226	40
21	9·162025	14·33	9·995372	·31	9·166654	14·64	10·833346	39
22	162885	14·30	995353	·31	167532	14·61	832468	38
23	163743	14·27	995334	·31	168409	14·58	831591	37
24	164600	14·24	995316	·31	169284	14·55	830716	36
25	165454	14·22	995297	·31	170157	14·53	829843	35
26	166307	14·19	995278	·31	171029	14·50	828971	34
27	167159	14·16	995260	·31	171899	14·47	828101	33
28	168008	14·13	995241	·32	172767	14·44	827233	32
29	168856	14·10	995222	·32	173634	14·42	826366	31
30	169702	14·07	995203	·32	174499	14·39	825501	30
31	9·170547	14·05	9·995184	·32	9·175362	14·36	10·824638	29
32	171389	14·02	995165	·32	176224	14·33	823776	28
33	172230	13·99	995146	·32	177084	14·31	822916	27
34	173070	13·96	995127	·32	177942	14·28	822058	26
35	173908	13·94	995108	·32	178799	14·25	821201	25
36	174744	13·91	995089	·32	179655	14·23	820345	24
37	175578	13·88	995070	·32	180508	14·20	819492	23
38	176411	13·86	995051	·32	181360	14·17	818640	22
39	177242	13·83	995032	·32	182211	14·15	817789	21
40	178072	13·80	995013	·32	183059	14·12	816941	20
41	9·178900	13·77	9·994993	·32	9·183907	14·09	10·816093	19
42	179726	13·74	994974	·32	184752	14·07	815248	18
43	180551	13·72	994955	·32	185597	14·04	814403	17
44	181374	13·69	994935	·32	186439	14·02	813561	16
45	182196	13·66	994916	·33	187280	13·99	812720	15
46	183016	13·64	994896	·33	188120	13·96	811880	14
47	183834	13·61	994877	·33	188958	13·93	811042	13
48	184651	13·59	994857	·33	189794	13·91	810206	12
49	185466	13·56	994838	·33	190629	13·89	809371	11
50	186280	13·53	994818	·33	191462	13·86	808538	10
51	9·187092	13·51	9·994798	·33	9·192294	13·84	10·807706	9
52	187903	13·48	994779	·33	193124	13·81	806876	8
53	188712	13·46	994759	·33	193953	13·79	806047	7
54	189519	13·43	994739	·33	194780	13·76	805220	6
55	190325	13·41	994719	·33	195606	13·74	804394	5
56	191130	13·38	994700	·33	196430	13·71	803570	4
57	191933	13·36	994680	·33	197253	13·69	802747	3
58	192734	13·33	994660	·33	198074	13·66	801926	2
59	193534	13·30	994640	·33	198894	13·64	801106	1
60	194332	13·28	994620	·33	199713	13·61	800287	0
	Cosine	D.	Sine	81°	Cotang.	D.	Tang.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9·194332	13·28	9·994620	·33	9·199713	13·61	10·800287	60
1	195129	13·26	994600	·33	200529	13·59	799471	59
2	195925	13·23	994580	·33	201345	13·56	798655	58
3	196719	13·21	994560	·34	202159	13·54	797841	57
4	197511	13·18	994540	·34	202971	13·52	797029	56
5	198302	13·16	994519	·34	203782	13·49	796218	55
6	199091	13·13	994499	·34	204592	13·47	795408	54
7	199879	13·11	994479	·34	205400	13·45	794600	53
8	200666	13·08	994459	·34	206207	13·42	793793	52
9	201451	13·06	994438	·34	207013	13·40	792987	51
10	202234	13·04	994418	·34	207817	13·38	792183	50
11	9·203017	13·01	9·994397	·34	9·208619	13·35	10·791381	49
12	203797	12·99	994377	·34	209420	13·33	790580	48
13	204577	12·96	994357	·34	210220	13·31	789780	47
14	205354	12·94	994336	·34	211018	13·28	788982	46
15	206131	12·92	994316	·34	211815	13·26	788185	45
16	206906	12·89	994295	·34	212611	13·24	787389	44
17	207679	12·87	994274	·35	213405	13·21	786595	43
18	208452	12·85	994254	·35	214198	13·19	785802	42
19	209222	12·82	994233	·35	214989	13·17	785011	41
20	209992	12·80	994212	·35	215780	13·15	784220	40
21	9·210760	12·78	9·994191	·35	9·216568	13·12	10·783432	39
22	211526	12·75	994171	·35	217356	13·10	782644	38
23	212291	12·73	994150	·35	218142	13·08	781858	37
24	213055	12·71	994129	·35	218926	13·05	781074	36
25	213818	12·68	994108	·35	219710	13·03	780290	35
26	214579	12·66	994087	·35	220492	13·01	779508	34
27	215338	12·64	994066	·35	221272	12·99	778728	33
28	216097	12·61	994045	·35	222052	12·97	777948	32
29	216854	12·59	994024	·35	222830	12·94	777170	31
30	217609	12·57	994003	·35	223606	12·92	776394	30
31	9·218363	12·55	9·993981	·35	9·224382	12·90	10·775618	29
32	219116	12·53	993960	·35	225156	12·88	774844	28
33	219868	12·50	993939	·35	225929	12·86	774071	27
34	220618	12·48	993918	·35	226700	12·84	773300	26
35	221367	12·46	993896	·36	227471	12·81	772529	25
36	222115	12·44	993875	·36	228239	12·79	771761	24
37	222861	12·42	993854	·36	229007	12·77	770993	23
38	223606	12·39	993832	·36	229773	12·75	770227	22
39	224349	12·37	993811	·36	230539	12·73	769461	21
40	225092	12·35	993789	·36	231302	12·71	768698	20
41	9·225833	12·33	9·993768	·36	9·232065	12·69	10·767935	19
42	226573	12·31	993746	·36	232826	12·67	767174	18
43	227311	12·28	993725	·36	233586	12·65	766414	17
44	228048	12·26	993703	·36	234345	12·62	765655	16
45	228784	12·24	993681	·36	235103	12·60	764897	15
46	229518	12·22	993660	·36	235859	12·58	764141	14
47	230252	12·20	993638	·36	236614	12·56	763386	13
48	230984	12·18	993616	·36	237368	12·54	762632	12
49	231714	12·16	993594	·37	238120	12·52	761880	11
50	232444	12·14	993572	·37	238872	12·50	761128	10
51	9·233172	12·12	9·993550	·37	9·239622	12·48	10·760378	9
52	233899	12·09	993528	·37	240371	12·46	759629	8
53	234625	12·07	993506	·37	241118	12·44	758882	7
54	235349	12·05	993484	·37	241865	12·42	758135	6
55	236073	12·03	993462	·37	242610	12·40	757390	5
56	236795	12·01	993440	·37	243354	12·38	756646	4
57	237515	11·99	993418	·37	244097	12·36	755903	3
58	238235	11·97	993396	·37	244839	12·34	755161	2
59	238953	11·95	993374	·37	245579	12·32	754421	1
60	239670	11·93	993351	·37	246319	12·30	753681	0
	Cosine	D.	Sine	80°	Cotang.	D.	Tang.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9·239670	11·93	9·993351	·37	9·246319	12·30	10·753681	60
1	240386	11·91	993329	·37	247057	12·28	752943	59
2	241101	11·89	993307	·37	247794	12·26	752206	58
3	241814	11·87	993285	·37	248530	12·24	751470	57
4	242526	11·85	993262	·37	249264	12·22	750736	56
5	243237	11·83	993240	·37	249998	12·20	750002	55
6	243947	11·81	993217	·38	250730	12·18	749270	54
7	244656	11·79	993195	·38	251461	12·17	748539	53
8	245363	11·77	993172	·38	252191	12·15	747809	52
9	246069	11·75	993149	·38	252920	12·13	747080	51
10	246775	11·73	993127	·38	253648	12·11	746352	50
11	9·247478	11·71	9·993104	·38	9·254374	12·09	10·745626	49
12	248181	11·69	993081	·38	255100	12·07	744900	48
13	248883	11·67	993059	·38	255824	12·05	744176	47
14	249583	11·65	993036	·38	256547	12·03	743453	46
15	2502[illegible]2	11·63	993013	·38	257269	12·01	742731	45
16	250980	11·61	992990	·38	257990	12·00	742010	44
17	251677	11·59	992967	·38	258710	11·98	741290	43
18	252373	11·58	992944	·38	259429	11·96	740571	42
19	253067	11·56	992921	·38	260146	11·94	739854	41
20	253761	11·54	992898	·38	260863	11·92	739137	40
21	9·254453	11·52	9·992875	·38	9·261578	11·90	10·738422	39
22	255144	11·50	992852	·38	262292	11·89	737708	38
23	255834	11·48	992829	·39	263005	11·87	736995	37
24	256523	11·46	992806	·39	263717	11·85	736283	36
25	257211	11·44	992783	·39	264428	11·83	735572	35
26	257898	11·42	992759	·39	265138	11·81	734862	34
27	258583	11·41	992736	·39	265847	11·79	734153	33
28	259268	11·39	992713	·39	266555	11·78	733445	32
29	259951	11·37	992690	·39	267261	11·76	732739	31
30	260633	11·35	992666	·39	267967	11·74	732033	30
31	9·261314	11·33	9·992643	·39	9·268671	11·72	10·731329	29
32	261994	11·31	992619	·39	269375	11·70	730625	28
33	262673	11·30	992596	·39	270077	11·69	729923	27
34	263351	11·28	992572	·39	270779	11·67	729221	26
35	264027	11·26	992549	·39	271479	11·65	728521	25
36	264703	11·24	992525	·39	272178	11·64	727822	24
37	265377	11·22	992501	·39	272876	11·62	727124	23
38	266051	11·20	992478	·40	273573	11·60	726427	22
39	266723	11·19	992454	·40	274269	11·58	725731	21
40	267395	11·17	992430	·40	274964	11·57	725036	20
41	9·268065	11·15	9·992406	·40	9·275658	11·55	10·724342	19
42	268734	11·13	992382	·40	276351	11·53	723649	18
43	269402	11·11	992359	·40	277043	11·51	722957	17
44	270069	11·10	992335	·40	277734	11·50	722266	16
45	270735	11·08	992311	·40	278424	11·48	721576	15
46	271400	11·06	992287	·40	279113	11·47	720887	14
47	272064	11·05	992263	·40	279801	11·45	720199	13
48	272726	11·03	992239	·40	280488	11·43	719512	12
49	273388	11·01	992214	·40	281174	11·41	718826	11
50	274049	10·99	992190	·40	281858	11·40	718142	10
51	9·274708	10·98	9·992166	·40	9·282542	11·38	10·717458	9
52	275367	10·96	992142	·40	283225	11·36	716775	8
53	276024	10·94	992117	·41	283907	11·35	716093	7
54	276681	10·92	992093	·41	284588	11·33	715412	6
55	277337	10·91	992069	·41	285268	11·31	714732	5
56	277991	10·89	992044	·41	285947	11·30	714053	4
57	278644	10·87	992020	·41	286624	11·28	713376	3
58	279297	10·86	991996	·41	287301	11·26	712699	2
59	279948	10·84	991971	·41	287977	11·25	712023	1
60	280599	10·82	991947	·41	288652	11·23	711348	0
	Cosine	D.	Sine	79°	Cotang.	D.	Tang.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9·280599	10·82	9·991947	·41	9·288652	11·23	10·711348	60
1	281248	10·81	991922	·41	289326	11·22	710674	59
2	281897	10·79	991897	·41	289999	11·20	710001	58
3	282544	10·77	991873	·41	290671	11·18	709329	57
4	283190	10·76	991848	·41	291342	11·17	708658	56
5	283836	10·74	991823	·41	292013	11·15	707987	55
6	284480	10·72	991799	·41	292682	11·14	707318	54
7	285124	10·71	991774	·42	293350	11·12	706650	53
8	285766	10·69	991749	·42	294017	11·11	705983	52
9	286408	10·67	991724	·42	294684	11·09	705316	51
10	287048	10·66	991699	·42	295349	11·07	704651	50
11	9·287687	10·64	9·991674	·42	9·296013	11·06	10·703987	49
12	288326	10·63	991649	·42	296677	11·04	703323	48
13	288964	10·61	991624	·42	297339	11·03	702661	47
14	289600	10·59	991599	·42	298001	11·01	701999	46
15	290236	10·58	991574	·42	298662	11·00	701338	45
16	290870	10·56	991549	·42	299322	10·98	700678	44
17	291504	10·54	991524	·42	299980	10·96	700020	43
18	292137	10·53	991498	·42	300638	10·95	699362	42
19	292768	10·51	991473	·42	301295	10·93	698705	41
20	293399	10·50	991448	·42	301951	10·92	698049	40
21	9·294029	10·48	9·991422	·42	9·302607	10·90	10·697393	39
22	294658	10·46	991397	·42	303261	10·89	696739	38
23	295286	10·45	991372	·43	303914	10·87	696086	37
24	295913	10·43	991346	·43	304567	10·86	695433	36
25	296539	10·42	991321	·43	305218	10·84	694782	35
26	297164	10·40	991295	·43	305869	10·83	694131	34
27	297788	10·39	991270	·43	306519	10·81	693481	33
28	298412	10·37	991244	·43	307168	10·80	692832	32
29	299034	10·36	991218	·43	307815	10·78	692185	31
30	299655	10·34	991193	·43	308463	10·77	691537	30
31	9·300276	10·32	9·991167	·43	9·309109	10·75	10·690891	29
32	300895	10·31	991141	·43	309754	10·74	690246	28
33	301514	10·29	991115	·43	310398	10·73	689602	27
34	302132	10·28	991090	·43	311042	10·71	688958	26
35	302748	10·26	991064	·43	311685	10·70	688315	25
36	303364	10·25	991038	·43	312327	10·68	687673	24
37	303979	10·23	991012	·43	312967	10·67	687033	23
38	304593	10·22	990986	·43	313608	10·65	686392	22
39	305207	10·20	990960	·43	314247	10·64	685753	21
40	305819	10·19	990934	·44	314885	10·62	685115	20
41	9·306430	10·17	9·990908	·44	9·315523	10·61	10·684477	19
42	307041	10·16	990882	·44	316159	10·60	683841	18
43	307650	10·14	990855	·44	316795	10·58	683205	17
44	308259	10·13	990829	·44	317430	10·57	682570	16
45	308867	10·11	990803	·44	318064	10·55	681936	15
46	309474	10·10	990777	·44	318697	10·54	681303	14
47	310080	10·08	990750	·44	319329	10·53	680671	13
48	310685	10·07	990724	·44	319961	10·51	680039	12
49	311289	10·05	990697	·44	320592	10·50	679408	11
50	311893	10·04	990671	·44	321222	10·48	678778	10
51	9·312495	10·03	9·990644	·44	9·321851	10·47	10·678149	9
52	313097	10·01	990618	·44	322479	10·45	677521	8
53	313698	10·00	990591	·44	323106	10·44	676894	7
54	314297	9·98	990565	·44	323733	10·43	676267	6
55	314897	9·97	990538	·44	324358	10·41	675642	5
56	315495	9·96	990511	·45	324983	10·40	675017	4
57	316092	9·94	990485	·45	325607	10·39	674393	3
58	316689	9·93	990458	·45	326231	10·37	673769	2
59	317284	9·91	990431	·45	326853	10·36	673147	1
60	317879	9·90	990404	·45	327475	10·35	672525	0
	Cosine	D.	Sine	78°	Cotang.	D.	Tang.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9·317879	9·90	9·990404	·45	9·327474	10·35	10·672526	60
1	318473	9·88	990378	·45	328095	10·33	671905	59
2	319066	9·87	990351	·45	328715	10·32	671285	58
3	319658	9·86	990324	·45	329334	10·30	670666	57
4	320249	9·84	990207	·45	329953	10·29	670047	56
5	320840	9·83	990270	·45	330570	10·28	669430	55
6	321430	9·82	990243	·45	331187	10·26	668813	54
7	322019	9·80	990215	·45	331803	10·25	668197	53
8	322607	9·79	990188	·45	332418	10·24	667582	52
9	323194	9·77	990161	·45	333033	10·23	666967	51
10	323780	9·76	990134	·45	333646	10·21	666354	50
11	9·324366	9·75	9·990107	·46	9·334259	10·20	10·665741	49
12	324950	9·73	990079	·46	334871	10·19	665129	48
13	325534	9·72	990052	·46	335482	10·17	664518	47
14	326117	9·70	990025	·46	336093	10·16	663907	46
15	326700	9·69	989997	·46	336702	10·15	663298	45
16	327281	9·68	989970	·46	337311	10·13	662689	44
17	327862	9·66	989942	·46	337919	10·12	662081	43
18	328442	9·65	989915	·46	338527	10·11	661473	42
19	329021	9·64	989887	·46	339133	10·10	660867	41
20	329599	9·62	989860	·46	339739	10·08	660261	40
21	9·330176	9·61	9·989832	·46	9·340344	10·07	10·659656	39
22	330753	9·60	989804	·46	340948	10·06	659052	38
23	331329	9·58	989777	·46	341552	10·04	658448	37
24	331903	9·57	989749	·47	342155	10·03	657845	36
25	332478	9·56	989721	·47	342757	10·02	657243	35
26	333051	9·54	989693	·47	343358	10·00	656642	34
27	333624	9·53	989665	·47	343958	9·99	656042	33
28	334195	9·52	989637	·47	344558	9·98	655442	32
29	334766	9·50	989609	·47	345157	9·97	654843	31
30	335337	9·49	989582	·47	345755	9·96	654245	30
31	9·335906	9·48	9·989553	·47	9·346353	9·94	10·653647	29
32	336475	9·46	989525	·47	346949	9·93	653051	28
33	337043	9·45	989497	·47	347545	9·92	652455	27
34	337610	9·44	989469	·47	348141	9·91	651859	26
35	338176	9·43	989441	·47	348735	9·90	651265	25
36	338742	9·41	989413	·47	349329	9·88	650671	24
37	339306	9·40	989384	·47	349922	9·87	650078	23
38	339871	9·39	989356	·47	350514	9·86	649486	22
39	340434	9·37	989328	·47	351106	9·85	648894	21
40	340996	9·36	989300	·47	351697	9·83	648303	20
41	9·341558	9·35	9·989271	·47	9·352287	9·82	10·647713	19
42	342119	9·34	989243	·47	352876	9·81	647124	18
43	342679	9·32	989214	·47	353465	9·80	646535	17
44	343239	9·31	989186	·47	354053	9·79	645947	16
45	343797	9·30	989157	·47	354640	9·77	645360	15
46	344355	9·29	989128	·48	355227	9·76	644773	14
47	344912	9·27	989100	·48	355813	9·75	644187	13
48	345469	9·26	989071	·48	356398	9·74	643602	12
49	346024	9·25	989042	·48	356982	9·73	643018	11
50	346579	9·24	989014	·48	357566	9·71	642434	10
51	9·347134	9·22	9·988985	·48	9·358149	9·70	10·641851	9
52	347687	9·21	988956	·48	358731	9·69	641269	8
53	348240	9·20	988927	·48	359313	9·68	640687	7
54	348792	9·19	988898	·48	359893	9·67	640107	6
55	349343	9·17	988869	·48	360474	9·66	639526	5
56	349893	9·16	988840	·48	361053	9·65	638947	4
57	350443	9·15	988811	·49	361632	9·63	638368	3
58	350992	9·14	988782	·49	362210	9·62	637790	2
59	351540	9·13	988753	·49	362787	9·61	637213	1
60	352088	9·11	988724	·49	363364	9·60	636636	0
	Cosine	D.	Sine	77°	Cotang.	D.	Tang.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9·352088	9·11	9·988724	·49	9·363364	9·60	10·636636	60
1	352635	9·10	988695	·49	363940	9·59	636060	59
2	353181	9·09	988666	·49	364515	9·58	635485	58
3	353726	9·08	988636	·49	365090	9·57	634910	57
4	354271	9·07	988607	·49	365664	9·55	634336	56
5	354815	9·05	988578	·49	366237	9·54	633763	55
6	355358	9·04	988548	·49	366810	9·53	633190	54
7	355901	9·03	988519	·49	367382	9·52	632618	53
8	356443	9·02	988489	·49	367953	9·51	632047	52
9	356984	9·01	988460	·49	368524	9·50	631476	51
10	357524	8·99	988430	·49	369094	9·49	630906	50
11	9·358064	8·98	9·988401	·49	9·369663	9·48	10·630337	49
12	358603	8·97	988371	·49	370232	9·46	629768	48
13	359141	8·96	988342	·49	370799	9·45	629201	47
14	359678	8·95	988312	·50	371367	9·44	628633	46
15	360215	8·93	988282	·50	371933	9·43	628067	45
16	360752	8·92	988252	·50	372499	9·42	627501	44
17	361287	8·91	988223	·50	373064	9·41	626936	43
18	361822	8·90	988193	·50	373629	9·40	626371	42
19	362356	8·89	988163	·50	374193	9·39	625807	41
20	362889	8·88	988133	·50	374756	9·38	625244	40
21	9·363422	8·87	9·988103	·50	9·375319	9·37	10·624681	39
22	363954	8·85	988073	·50	375881	9·35	624119	38
23	364485	8·84	988043	·50	376442	9·34	623558	37
24	365016	8·83	988013	·50	377003	9·33	622997	36
25	365546	8·82	987983	·50	377563	9·32	622437	35
26	366075	8·81	987953	·50	378122	9·31	621878	34
27	366604	8·80	987922	·50	378681	9·30	621319	33
28	367131	8·79	987892	·50	379239	9·29	620761	32
29	367659	8·77	987862	·50	379797	9·28	620203	31
30	368185	8·76	987832	·51	380354	9·27	619646	30
31	9·368711	8·75	9·987801	·51	9·380910	9·26	10·619090	29
32	369236	8·74	987771	·51	381466	9·25	618534	28
33	369761	8·73	987740	·51	382020	9·24	617980	27
34	370285	8·72	987710	·51	382575	9·23	617425	26
35	370808	8·71	987679	·51	383129	9·22	616871	25
36	371330	8·70	987649	·51	383682	9·21	616318	24
37	371852	8·69	987618	·51	384234	9·20	615766	23
38	372373	8·67	987588	·51	384786	9·19	615214	22
39	372894	8·66	987557	·51	385337	9·18	614663	21
40	373414	8·65	987526	·51	385888	9·17	614112	20
41	9·373933	8·64	9·987496	·51	9·386438	9·15	10·613562	19
42	374452	8·63	987465	·51	386987	9·14	613013	18
43	374970	8·62	987434	·51	387536	9·13	612464	17
44	375487	8·61	987403	·52	388084	9·12	611916	16
45	376003	8·60	987372	·52	388631	9·11	611369	15
46	376519	8·59	987341	·52	389178	9·10	610822	14
47	377035	8·58	987310	·52	389724	9·09	610276	13
48	377549	8·57	987279	·52	390270	9·08	609730	12
49	378063	8·56	987248	·52	390815	9·07	609185	11
50	378577	8·54	987217	·52	391360	9·06	608640	10
51	9·379089	8·53	9·987186	·52	9·391903	9·05	10·608097	9
52	379601	8·52	987155	·52	392447	9·04	607553	8
53	380113	8·51	987124	·52	392989	9·03	607011	7
54	380624	8·50	987092	·52	393531	9·02	606469	6
55	381134	8·49	987061	·52	394073	9·01	605927	5
56	381643	8·48	987030	·52	394614	9·00	605386	4
57	382152	8·47	986998	·52	395154	8·99	604846	3
58	382661	8·46	986967	·52	395694	8·98	604306	2
59	383168	8·45	986936	·52	396233	8·97	603767	1
60	383675	8·44	986904	·52	396771	8·96	603229	0
	Cosine	D.	Sine	76°	Cotang.	D.	Tang.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9·383675	8·44	9·986904	·52	9·396771	8·96	10·603229	60
1	384182	8·43	986873	·53	397309	8·96	602691	59
2	384687	8·42	986841	·53	397846	8·95	602154	58
3	385192	8·41	986809	·53	398383	8·94	601617	57
4	385697	8·40	986778	·53	398919	8·93	601081	56
5	386201	8·39	986746	·53	399455	8·92	600545	55
6	386704	8·38	986714	·53	399990	8·91	600010	54
7	387207	8·37	986683	·53	400524	8·90	599476	53
8	387709	8·36	986651	·53	401058	8·89	598942	52
9	388210	8·35	986619	·53	401591	8·88	598409	51
10	388711	8·34	986587	·53	402124	8·87	597876	50
11	9·389211	8·33	9·986555	·53	9·402656	8·86	10·597344	49
12	389711	8·32	986523	·53	403187	8·85	596813	48
13	390210	8·31	986491	·53	403718	8·84	596282	47
14	390708	8·30	986459	·53	404249	8·83	595751	46
15	391206	8·28	86427	·53	404778	8·82	595222	45
16	391703	8·27	986395	·53	405308	8·81	594692	44
17	392199	8·26	986363	·54	405836	8·80	594164	43
18	392695	8·25	986331	·54	406364	8·79	593636	42
19	393191	8·24	986299	·54	406892	8·78	593108	41
20	393685	8·23	986266	·54	407419	8·77	592581	40
21	9·394179	8·22	9·986234	·54	9·407945	8·76	10·592055	39
22	394673	8·21	986202	·54	408471	8·75	591529	38
23	395166	8·20	986169	·54	408997	8·74	591003	37
24	395658	8·19	986137	·54	409521	8·74	590479	36
25	396150	8·18	986104	·54	410045	8·73	589955	35
26	396641	8·17	986072	·54	410569	8·72	589431	34
27	397132	8·17	986039	·54	411092	8·71	588908	33
28	397621	8·16	986007	·54	411615	8·70	588385	32
29	398111	8·15	985974	·54	412137	8·69	587863	31
30	398600	8·14	985942	·54	412658	8·68	587342	30
31	9·399088	8·13	9·985909	·55	9·413179	8·67	10·586821	29
32	399575	8·12	985876	·55	413699	8·66	586301	28
33	400062	8·11	985843	·55	414219	8·65	585781	27
34	400549	8·10	985811	·55	414738	8·64	585262	26
35	401035	8·09	985778	·55	415257	8·64	584743	25
36	401520	8·08	985745	·55	415775	8·63	584225	24
37	402005	8·07	985712	·55	416293	8·62	583707	23
38	402489	8·06	985679	·55	416810	8·61	583190	22
39	402972	8·05	985646	·55	417326	8·60	582674	21
40	403455	8·04	985613	·55	417842	8·59	582158	20
41	9·403938	8·03	9·985580	·55	9·418358	8·58	10·581642	19
42	404420	8·02	985547	·55	418873	8·57	581127	18
43	404901	8·01	985514	·55	419387	8·56	580613	17
44	405382	8·00	985480	·55	419901	8·55	580099	16
45	405862	7·99	985447	·55	420415	8·55	579585	15
46	406341	7·98	985414	·56	420927	8·54	579073	14
47	406820	7·97	985380	·56	421440	8·53	578560	13
48	407299	7·96	985347	·56	421952	8·52	578048	12
49	407777	7·95	985314	·56	422463	8·51	577537	11
50	408254	7·94	985280	·56	422974	8·50	577026	10
51	9·408731	7·94	9·985247	·56	9·423484	8·49	10·576516	9
52	409207	7·93	985213	·56	423993	8·48	576007	8
53	409682	7·92	985180	·56	424503	8·48	575497	7
54	410157	7·91	985146	·56	425011	8·47	574989	6
55	410632	7·90	985113	·56	425519	8·46	574481	5
56	411106	7·89	985079	·56	426027	8·45	573973	4
57	411579	7·88	985045	·56	426534	8·44	573466	3
58	412052	7·87	985011	·56	427041	8·43	572959	2
59	412524	7·86	984978	·56	427547	8·43	572453	1
60	412996	7·85	984944	·56	428052	8·42	571948	0
	Cosine	D.	Sine	75°	Cotang.	D.	Tang.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9·412996	7·85	9·984944	·57	9·428052	8·42	10·571948	60
1	413467	7·84	984910	·57	428557	8·41	571443	59
2	413938	7·83	984876	·57	429062	8·40	570938	58
3	414408	7·83	984842	·57	429566	8·39	570434	57
4	414878	7·82	984808	·57	430070	8·38	569930	56
5	415347	7·81	984774	·57	430573	8·38	569427	55
6	415815	7·80	984740	·57	431075	8·37	568925	54
7	416283	7·79	984706	·57	431577	8·36	568423	53
8	416751	7·78	984672	·57	432079	8·35	567921	52
9	417217	7·77	984637	·57	432580	8·34	567420	51
10	417684	7·76	984603	·57	433080	8·33	566920	50
11	9·418150	7·75	9·984569	·57	9·433580	8·32	10·566420	49
12	418615	7·74	984535	·57	434080	8·32	565920	48
13	419079	7·73	984500	·57	434579	8·31	565421	47
14	419544	7·73	984466	·57	435078	8·30	564922	46
15	420007	7·72	984432	·58	435576	8·29	564424	45
16	420470	7·71	984397	·58	436073	8·28	563927	44
17	420933	7·70	984363	·58	436570	8·28	563430	43
18	421395	7·69	984328	·58	437067	8·27	562933	42
19	421857	7·68	984294	·58	437563	8·26	562437	41
20	422318	7·67	984259	·58	438059	8·25	561941	40
21	9·422778	7·67	9·984224	·58	9·438554	8·24	10·561446	39
22	423238	7·66	984190	·58	439048	8·23	560952	38
23	423697	7·65	984155	·58	439543	8·23	560457	37
24	424156	7·64	984120	·58	440036	8·22	559964	36
25	424615	7·63	984085	·58	440529	8·21	559471	35
26	425073	7·62	984050	·58	441022	8·20	558978	34
27	425530	7·61	984015	·58	441514	8·19	558486	33
28	425987	7·60	983981	·58	442006	8·19	557994	32
29	426443	7·60	983946	·58	442497	8·18	557503	31
30	426899	7·59	983911	·58	442988	8·17	557012	30
31	9·427354	7·58	9·983875	·58	9·443479	8·16	10·556521	29
32	427809	7·57	983840	·59	443968	8·16	556032	28
33	428263	7·56	983805	·59	444458	8·15	555542	27
34	428717	7·55	983770	·59	444947	8·14	555053	26
35	429170	7·54	983735	·59	445435	8·13	554565	25
36	429623	7·53	983700	·59	445923	8·12	554077	24
37	430075	7·52	983664	·59	446411	8·12	553589	23
38	430527	7·52	983629	·59	446898	8·11	553102	22
39	430978	7·51	983594	·59	447384	8·10	552616	21
40	431429	7·50	983558	·59	447870	8·09	552130	20
41	9·431879	7·49	9·983523	·59	9·448356	8·09	10·551644	19
42	432329	7·49	983487	·59	448841	8·08	551159	18
43	432778	7·48	983452	·59	449326	8·07	550674	17
44	433226	7·47	983416	·59	449810	8·06	550190	16
45	433675	7·46	983381	·59	450294	8·06	549706	15
46	434122	7·45	983345	·59	450777	8·05	549223	14
47	434569	7·44	983309	·59	451260	8·04	548740	13
48	435016	7·44	983273	·60	451743	8·03	548257	12
49	435462	7·43	983238	·60	452225	8·02	547775	11
50	435908	7·42	983202	·60	452706	8·02	547294	10
51	9·436353	7·41	9·983166	·60	9·453187	8·01	10·546813	9
52	436798	7·40	983130	·60	453668	8·00	546332	8
53	437242	7·40	983094	·60	454148	7·99	545852	7
54	437686	7·39	983058	·60	454628	7·99	545372	6
55	438129	7·38	983022	·60	455107	7·98	544893	5
56	438572	7·37	982986	·60	455586	7·97	544414	4
57	439014	7·36	982950	·60	456064	7·96	543936	3
58	439456	7·36	982914	·60	456542	7·96	543458	2
59	439897	7·35	982878	·60	457019	7·95	542981	1
60	440338	7·34	982842	·60	457496	7·94	542504	0
	Cosine	D.	Sine	74°	Cotang.	D.	Tang.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9·440338	7·34	9·982842	·60	9·457496	7·94	10·542504	60
1	440778	7·33	982805	·60	457973	7·93	542027	59
2	441218	7·32	982769	·61	458449	7·93	541551	58
3	441658	7·31	982733	·61	458925	7·92	541075	57
4	442096	7·31	982696	·61	459400	7·91	540600	56
5	442535	7·30	982660	·61	459875	7·90	540125	55
6	442973	7·29	982624	·61	460349	7·90	539651	54
7	443410	7·28	982587	·61	460823	7·89	539177	53
8	443847	7·27	982551	·61	461297	7·88	538703	52
9	444284	7·27	982514	·61	461770	7·88	538230	51
10	444720	7·26	982477	·61	462242	7·87	537758	50
11	9·445155	7·25	9·982441	·61	9·462714	7·86	10·537286	49
12	445590	7·24	982404	·61	463186	7·85	536814	48
13	446025	7·23	982367	·61	463658	7·85	536342	47
14	446459	7·23	982331	·61	464129	7·84	535871	46
15	446893	7·22	982294	·61	464599	7·83	535401	45
16	447326	7·21	982257	·61	465069	7·83	534931	44
17	447759	7·20	982220	·62	465539	7·82	534461	43
18	448191	7·20	982183	·62	466008	7·81	533992	42
19	448623	7·19	982146	·62	466476	7·80	533524	41
20	449054	7·18	982109	·62	466945	7·80	533055	40
21	9·449485	7·17	9·982072	·62	9·467413	7·79	10·532587	39
22	449915	7·16	982035	·62	467880	7·78	532120	38
23	450345	7·16	981998	·62	468347	7·78	531653	37
24	450775	7·15	981961	·62	468814	7·77	531186	36
25	451204	7·14	981924	·62	469280	7·76	530720	35
26	451632	7·13	981886	·62	469746	7·75	530254	34
27	452060	7·13	981849	·62	470211	7·75	529789	33
28	452488	7·12	981812	·62	470676	7·74	529324	32
29	452915	7·11	981774	·62	471141	7·73	528859	31
30	453342	7·10	981737	·62	471605	7·73	528395	30
31	9·453768	7·10	9·981699	·63	9·472068	7·72	10·527932	29
32	454194	7·09	981662	·63	472532	7·71	527468	28
33	454619	7·08	981625	·63	472995	7·71	527005	27
34	455044	7·07	981587	·63	473457	7·70	526543	26
35	455469	7·07	981549	·63	473919	7·69	526081	25
36	455893	7·06	981512	·63	474381	7·69	525619	24
37	456316	7·05	981474	·63	474842	7·68	525158	23
38	456739	7·04	981436	·63	475303	7·67	524697	22
39	457162	7·04	981399	·63	475763	7·67	524237	21
40	457584	7·03	981361	·63	476223	7·66	523777	20
41	9·458006	7·02	9·981323	·63	9·476683	7·65	10·523317	19
42	458427	7·01	981285	·63	477142	7·65	522858	18
43	458848	7·01	981247	·63	477601	7·64	522399	17
44	459268	7·00	981209	·63	478059	7·63	521941	16
45	459688	6·99	981171	·63	478517	7·63	521483	15
46	460108	6·98	981133	·64	478975	7·62	521025	14
47	460527	6·98	981095	·64	479432	7·61	520568	13
48	460946	6·97	981057	·64	479889	7·61	520111	12
49	461364	6·96	981019	·64	480345	7·60	519655	11
50	461782	6·95	980981	·64	480801	7·59	519199	10
51	9·462199	6·95	9·980942	·64	9·481257	7·59	10·518743	9
52	462616	6·94	980904	·64	481712	7·58	518288	8
53	463032	6·93	980866	·64	482167	7·57	517833	7
54	463448	6·93	980827	·64	482621	7·57	517379	6
55	463864	6·92	980789	·64	483075	7·56	516925	5
56	464279	6·91	980750	·64	483529	7·55	516471	4
57	464694	6·90	980712	·64	483982	7·55	516018	3
58	465108	6·90	980673	·64	484435	7·54	515565	2
59	465522	6·89	980635	·64	484887	7·53	515113	1
60	465935	6·88	980596	·64	485339	7·53	514661	0
	Cosine	D.	Sine	73°	Cotang.	D.	Tang.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9·465935	6·88	9·980596	·64	9·485339	7·55	10·514661	60
1	466348	6·88	980558	·64	485791	7·52	514209	59
2	466761	6·87	980519	·65	486242	7·51	513758	58
3	467173	6·86	980480	·65	486693	7·51	513307	57
4	467585	6·85	980442	·65	487143	7·50	512857	56
5	467996	6·85	980403	·65	487593	7·49	512407	55
6	468407	6·84	980364	·65	488043	7·49	511957	54
7	468817	6·83	980325	·65	488492	7·48	511508	53
8	469227	6·83	980286	·65	488941	7·47	511059	52
9	469637	6·82	980247	·65	489390	7·47	510610	51
10	470046	6·81	980208	·65	489838	7·46	510162	50
11	9·470455	6·80	9·980169	·65	9·490286	7·46	10·509714	49
12	470863	6·80	980130	·65	490733	7·45	509267	48
13	471271	6·79	980091	·65	491180	7·44	508820	47
14	471679	6·78	980052	·65	491627	7·44	508373	46
15	472086	6·78	980012	·65	492073	7·43	507927	45
16	472492	6·77	979973	·65	492519	7·43	507481	44
17	472898	6·76	979934	·66	492965	7·42	507035	43
18	473304	6·76	979895	·66	493410	7·41	506590	42
19	473710	6·75	979855	·66	493854	7·40	506146	41
20	474115	6·74	979816	·66	494299	7·40	505701	40
21	9·474519	6·74	9·979776	·66	9·494743	7·40	10·505257	39
22	474923	6·73	979737	·66	495186	7·39	504814	38
23	475327	6·72	979697	·66	495630	7·38	504370	37
24	475730	6·72	979658	·66	496073	7·37	503927	36
25	476133	6·71	979618	·66	496515	7·37	503485	35
26	476536	6·70	979579	·66	496957	7·36	503043	34
27	476938	6·69	979539	·66	497399	7·36	502601	33
28	477340	6·69	979499	·66	497841	7·35	502159	32
29	477741	6·68	979459	·66	498282	7·34	501718	31
30	478142	6·67	979420	·66	498722	7·34	501278	30
31	9·478542	6·67	9·979380	·66	9·499163	7·33	10·500837	29
32	478942	6·66	979340	·66	499603	7·33	500397	28
33	479342	6·65	979300	·67	500042	7·32	499958	27
34	479741	6·65	979260	·67	500481	7·31	499519	26
35	480140	6·64	979220	·67	500920	7·31	499080	25
36	480539	6·63	979180	·67	501359	7·30	498641	24
37	480937	6·63	979140	·67	501797	7·30	498203	23
38	481334	6·62	979100	·67	502235	7·29	497765	22
39	481731	6·61	979059	·67	502672	7·28	497328	21
40	482128	6·61	979019	·67	503109	7·28	496891	20
41	9·482525	6·60	9·978979	·67	9·503546	7·27	10·496454	19
42	482921	6·59	978939	·67	503982	7·27	496018	18
43	483316	6·59	978898	·67	504418	7·26	495582	17
44	483712	6·58	978858	·67	504854	7·25	495146	16
45	484107	6·57	978817	·67	505289	7·25	494711	15
46	484501	6·57	978777	·67	505724	7·24	494276	14
47	484895	6·56	978736	·67	506159	7·24	493841	13
48	485289	6·55	978696	·68	506593	7·23	493407	12
49	485682	6·55	978655	·68	507027	7·22	492973	11
50	486075	6·54	978615	·68	507460	7·22	492540	10
51	9·486467	6·53	9·978574	·68	9·507893	7·21	10·492107	9
52	486860	6·53	978533	·68	508326	7·21	491674	8
53	487251	6·52	978493	·68	508759	7·20	491241	7
54	487643	6·51	978452	·68	509191	7·19	490809	6
55	488034	6·51	978411	·68	509622	7·19	490378	5
56	488424	6·50	978370	·68	510054	7·18	489946	4
57	488814	6·50	978329	·68	510485	7·18	489515	3
58	489204	6·49	978288	·68	510916	7·17	489084	2
59	489593	6·48	978247	·68	511346	7·16	488654	1
60	489982	6·48	978206	·68	511776	7·16	488224	0
	Cosine	D.	Sine	72°	Cotang.	D.	Tang.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9·489982	6·48	9·978206	·68	9·511776	7·16	10·488224	60
1	490371	6·48	978165	·68	512206	7·16	487794	59
2	490759	6·47	978124	·68	512635	7·15	487365	58
3	491147	6·46	978083	·69	513064	7·14	486936	57
4	491535	6·46	978042	·69	513493	7·14	486507	56
5	491922	6·45	978001	·69	513921	7·13	486079	55
6	492308	6·44	977959	·69	514349	7·13	485651	54
7	492695	6·44	977918	·69	514777	7·12	485223	53
8	493081	6·43	977877	·69	515204	7·12	484796	52
9	493466	6·42	977835	·69	515631	7·11	484369	51
10	493851	6·42	977794	·69	516057	7·10	483943	50
11	9·494236	6·41	9·977752	·69	9·516484	7·10	10·483516	49
12	494621	6·41	977711	·69	516910	7·09	483090	48
13	495005	6·40	977669	·69	517335	7·09	482665	47
14	495388	6·39	977628	·69	517761	7·08	482239	46
15	495772	6·39	977586	·69	518185	7·08	481815	45
16	496154	6·38	977544	·70	518610	7·07	481390	44
17	496537	6·37	977503	·70	519034	7·06	480966	43
18	496919	6·37	977461	·70	519458	7·06	480542	42
19	497301	6·36	977419	·70	519882	7·05	480118	41
20	497682	6·36	977377	·70	520305	7·05	479695	40
21	9·498064	6·35	9·977335	·70	9·520728	7·04	10·479272	39
22	498444	6·34	977293	·70	521151	7·03	478849	38
23	498825	6·34	977251	·70	521573	7·03	478427	37
24	499204	6·33	977209	·70	521995	7·03	478005	36
25	499584	6·32	977167	·70	522417	7·02	477583	35
26	499963	6·32	977125	·70	522838	7·02	477162	34
27	500342	6·31	977083	·70	523259	7·01	476741	33
28	500721	6·31	977041	·70	523680	7·01	476320	32
29	501099	6·30	976999	·70	524100	7·00	475900	31
30	501476	6·29	976957	·70	524520	6·99	475480	30
31	9·501854	6·29	9·976914	·70	9·524939	6·99	10·475061	29
32	502231	6·28	976872	·71	525359	6·98	474641	28
33	502607	6·28	976830	·71	525778	6·98	474222	27
34	502984	6·27	976787	·71	526197	6·97	473803	26
35	503360	6·26	976745	·71	526615	6·97	473385	25
36	503735	6·26	976702	·71	527033	6·96	472967	24
37	504110	6·25	976660	·71	527451	6·96	472549	23
38	504485	6·25	976617	·71	527868	6·95	472132	22
39	504860	6·24	976574	·71	528285	6·95	471715	21
40	505234	6·23	976532	·71	528702	6·94	471298	20
41	9·505608	6·23	9·976489	·71	9·529119	6·93	10·470881	19
42	505981	6·22	976446	·71	529535	6·93	470465	18
43	506354	6·22	976404	·71	529950	6·93	470050	17
44	506727	6·21	976361	·71	530366	6·92	469634	16
45	507099	6·20	676318	·71	530781	6·91	469219	15
46	507471	6·20	976275	·71	531196	6·91	468804	14
47	507843	6·19	976232	·72	531611	6·90	468389	13
48	508214	6·19	976189	·72	532025	6·90	467975	12
49	508585	6·18	976146	·72	532439	6·89	467561	11
50	508956	6·18	976103	·72	532853	6·89	467147	10
51	9·509326	6·17	9·976060	·72	9·533266	6·88	10·466734	9
52	509696	6·16	976017	·72	533679	6·88	466321	8
53	510065	6·16	975974	·72	534092	6·87	465908	7
54	510434	6·15	975930	·72	534504	6·87	465496	6
55	510803	6·15	975887	·72	534916	6·86	465084	5
56	511172	6·14	975844	·72	535328	6·86	464672	4
57	511540	6·13	975800	·72	535739	6·85	464261	3
58	511907	6·13	975757	·72	536150	6·S5	463850	2
59	512275	6·12	975714	·72	536561	6·84	463439	1
60	512642	6·12	975670	·72	536972	6·84	463028	0
	Cosine	D.	Sine	71°	Cotang.	D.	Tang.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9·512642	6·12	9·975670	·73	9·536972	6·84	10·463028	60
1	513009	6·11	975627	·73	537382	6·83	462618	59
2	513375	6·11	975583	·73	537792	6·83	462208	58
3	513741	6·10	975539	·73	538202	6·82	461798	57
4	514107	6·09	975496	·73	538611	6·82	461389	56
5	514472	6·09	975452	·73	539020	6·81	460980	55
6	514837	6·08	975408	·73	539429	6·81	460571	54
7	515202	6·08	975365	·73	539837	6·80	460163	53
8	515566	6·07	975321	·73	540245	6·80	459755	52
9	515930	6·07	975277	·73	540653	6·79	459347	51
10	516294	6·06	975233	·73	541061	6·79	458939	50
11	9·516657	6·05	9·975189	·73	9·541468	6·78	10·458532	49
12	517020	6·05	975145	·73	541875	6·78	458125	48
13	517382	6·04	975101	·73	542281	6·77	457719	47
14	517745	6·04	975057	·73	542688	6·77	457312	46
15	518107	6·03	975013	·73	543094	6·76	456906	45
16	518468	6·03	974969	·74	543499	6·76	456501	44
17	518829	6·02	974925	·74	543905	6·75	456095	43
18	519190	6·01	974880	·74	544310	6·75	455690	42
19	519551	6·01	974836	·74	544715	6·74	455285	41
20	519911	6·00	974792	·74	545119	6·74	454881	40
21	9·520271	6·00	9·974748	·74	9·545524	6·73	10·454476	39
22	520631	5·99	974703	·74	545928	6·73	454072	38
23	520990	5·99	974659	·74	546331	6·72	453669	37
24	521349	5·98	974614	·74	546735	6·72	453265	36
25	521707	5·98	974570	·74	547138	6·71	452862	35
26	522066	5·97	974525	·74	547540	6·71	452460	34
27	522424	5·96	974481	·74	547943	6·70	452057	33
28	522781	5·96	974436	·74	548345	6·70	451655	32
29	523138	5·95	974391	·74	548747	6·69	451253	31
30	523495	5·95	974347	·75	549149	6·69	450851	30
31	9·523852	5·94	9·974302	·75	9·549550	6·68	10·450450	29
32	524208	5·94	974257	·75	549951	6·68	450049	28
33	524564	5·93	974212	·75	550352	6·67	449648	27
34	524920	5·93	974167	·75	550752	6·67	449248	26
35	525275	5·92	974122	·75	551152	6·66	448848	25
36	525630	5·91	974077	·75	551552	6·66	448448	24
37	525984	5·91	974032	·75	551952	6·65	448048	23
38	526339	5·90	973987	·75	552351	6·65	447649	22
39	526693	5·90	973942	·75	552750	6·65	447250	21
40	527046	5·89	973897	·75	553149	6·64	446851	20
41	9·527400	5·89	9·973852	·75	9·553548	6·64	10·446452	19
42	527753	5·88	973807	·75	553946	6·63	446054	18
43	528105	5·88	973761	·75	554344	6·63	445656	17
44	528458	5·87	973716	·76	554741	6·62	445259	16
45	528810	5·87	973671	·76	555139	6·62	444861	15
46	529161	5·86	973625	·76	555536	6·61	444464	14
47	529513	5·86	973580	·76	555933	6·61	444067	13
48	529864	5·85	973535	·76	556329	6·60	443671	12
49	530215	5·85	973489	·76	556725	6·60	443275	11
50	530565	5·84	973444	·76	557121	6·59	442879	10
51	9·530915	5·84	9·973398	·76	9·557517	6·59	10·442483	9
52	531265	5·83	973352	·76	557913	6·59	442087	8
53	531614	5·82	973307	·76	558308	6·58	441692	7
54	531963	5·82	973261	·76	558702	6·58	441298	6
55	532312	5·81	973215	·76	559097	6·57	440903	5
56	532661	5·81	973169	·76	559491	6·57	440509	4
57	533009	5·80	973124	·76	559885	6·56	440115	3
58	533357	5·80	973078	·76	560279	6·56	439721	2
59	533704	5·79	973032	·77	560673	6·55	439327	1
60	534052	5·78	972986	·77	561066	6·55	438934	0
	Cosine	D.	Sine	70°	Cotang.	D.	Tang.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9·534052	5·78	9·972986	·77	9·561066	6·55	10·438934	60
1	534399	5·77	972940	·77	561459	6·54	438541	59
2	534745	5·77	972894	·77	561851	6·54	438149	58
3	535092	5·77	972848	·77	562244	6·53	437756	57
4	535438	5·76	972802	·77	562636	6·53	437364	56
5	535783	5·76	972755	·77	563028	6·53	436972	55
6	536129	5·75	972709	·77	563419	6·52	436581	54
7	536474	5·74	972663	·77	563811	6·52	436189	53
8	536818	5·74	972617	·77	564202	6·51	435798	52
9	537163	5·73	972570	·77	564592	6·51	435408	51
10	537507	5·73	972524	·77	564983	6·50	435017	50
11	9·537851	5·72	9·972478	·77	9·565373	6·50	10·434627	49
12	538194	5·72	972431	·78	565763	6·49	434237	48
13	538538	5·71	972385	·78	566153	6·49	433847	47
14	538880	5·71	972338	·78	566542	6·49	433458	46
15	539223	5·70	972291	·78	566932	6·48	433068	45
16	539565	5·70	972245	·78	567320	6·48	432680	44
17	539907	5·69	972198	·78	567709	6·47	432291	43
18	540249	5·69	972151	·78	568098	6·47	431902	42
19	540590	5·68	972105	·78	568486	6·46	431514	41
20	540931	5·68	972058	·78	568873	6·46	431127	40
21	9·541272	5·67	9·972011	·78	9·569261	6·45	10·430739	39
22	541613	5·67	971964	·78	569648	6·45	430352	38
23	541953	5·66	971917	·78	570035	6·45	429965	37
24	542293	5·66	971870	·78	570422	6·44	429578	36
25	542632	5·65	971823	·78	570809	6·44	429191	35
26	542971	5·65	971776	·78	571195	6·43	428805	34
27	543310	5·64	971729	·79	571581	6·43	428419	33
28	543649	5·64	971682	·79	571967	6·42	428033	32
29	543987	5·63	971635	·79	572352	6·42	427648	31
30	544325	5·63	971588	·79	572738	6·42	427262	30
31	9·544663	5·62	9·971540	·79	9·573123	6·41	10·426877	29
32	545000	5·62	971493	·79	573507	6·41	426493	28
33	545338	5·61	971446	·79	573892	6·40	426108	27
34	545674	5·61	971398	·79	574276	6·40	425724	26
35	546011	5·60	971351	·79	574660	6·39	425340	25
36	546347	5·60	971303	·79	575044	6·39	424956	24
37	546683	5·59	971256	·79	575427	6·39	424573	23
38	547019	5·59	971208	·79	575810	6·38	424190	22
39	547354	5·58	971161	·79	576193	6·38	423807	21
40	547689	5·58	971113	·79	576576	6·37	423424	20
41	9·548024	5·57	9·971066	·80	9·576958	6·37	10·423041	19
42	548359	5·57	971018	·80	577341	6·36	422659	18
43	548693	5·56	970970	·80	577723	6·36	422277	17
44	549027	5·56	970922	·80	578104	6·36	421896	16
45	549360	5·55	970874	·80	578486	6·35	421514	15
46	549693	5·55	970827	·80	578867	6·35	421133	14
47	550026	5·54	970779	·80	579248	6·34	420752	13
48	550359	5·54	970731	·80	579629	6·34	420371	12
49	550692	5·53	970683	·80	580009	6·34	419991	11
50	551024	5·53	970635	·80	580389	6·33	419611	10
51	9·551356	5·52	9·970586	·80	9·580769	6·33	10·419231	9
52	551687	5·52	970538	·80	581149	6·32	418851	8
53	552018	5·52	970490	·80	581528	6·32	418472	7
54	552349	5·51	970442	·80	581907	6·32	418093	6
55	552680	5·51	970394	·80	582286	6·31	417714	5
56	553010	5·50	970345	·81	582665	6·31	417335	4
57	553341	5·50	970297	·81	583043	6·30	416957	3
58	553670	5·49	970249	·81	583422	6·30	416578	2
59	554000	5·49	970200	·81	583800	6·29	416200	1
60	554329	5·48	970152	·81	584177	6·29	415823	0
	Cosine	D.	Sine	69°	Cotang.	D.	Tang.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9·554329	5·48	9·970152	·81	9·584177	6·29	10·415823	60
1	554658	5·48	970103	·81	584555	6·29	415445	59
2	554987	5·47	970055	·81	584932	6·28	415068	58
3	555315	5·47	970006	·81	585309	6·28	414691	57
4	555643	5·46	969957	·81	585686	6·27	414314	56
5	555971	5·46	969909	·81	586062	6·27	413938	55
6	556299	5·45	969860	·81	586439	6·27	413561	54
7	556626	5·45	969811	·81	586815	6·26	413185	53
8	556953	5·44	969762	·81	587190	6·26	412810	52
9	557280	5·44	969714	·81	587566	6·25	412434	51
10	557606	5·43	969665	·81	587941	6·25	412059	50
11	9·557932	5·43	9·969616	·82	9·588316	6·25	10·411684	49
12	558258	5·43	969567	·82	588691	6·24	411309	48
13	558583	5·42	969518	·82	589066	6·24	410934	47
14	558909	5·42	969469	·82	589440	6·23	410560	46
15	559234	5·41	969420	·82	589814	6·23	410186	45
16	559558	5·41	969370	·82	590188	6·23	409812	44
17	559883	5·40	969321	·82	590562	6·22	409438	43
18	560207	5·40	969272	·82	590935	6·22	409065	42
19	60531	5·39	969223	·82	591308	6·22	408692	41
20	560855	5·39	969173	·82	591681	6·21	408319	40
21	9·561178	5·38	9·969124	·82	9·592054	6·21	10·407946	39
22	561501	5·38	969075	·82	592426	6·20	407574	38
23	561824	5·37	969025	·82	592798	6·20	407202	37
24	562146	5·37	968976	·82	593170	6·19	406829	36
25	562468	5·36	968926	·83	593542	6·19	406458	35
26	562790	5·36	968877	·83	593914	6·18	406086	34
27	563112	5·36	968827	·83	594285	6·18	405715	33
28	563433	5·35	968777	·83	594656	6·18	405344	32
29	563755	5·35	968728	·83	595027	6·17	404973	31
30	564075	5·34	968678	·83	595398	6·17	404602	30
31	9·564396	5·34	9·968628	·83	9·595768	6·17	10·404232	29
32	564716	5·33	968578	·83	596138	6·16	403862	28
33	565036	5·33	968528	·83	596508	6·16	403492	27
34	565356	5·32	968479	·83	596878	6·16	403122	26
35	565676	5·32	968429	·83	597247	6·15	402753	25
36	565995	5·31	968379	·83	597616	6·15	402384	24
37	566314	5·31	968329	·83	597985	6·15	402015	23
38	566632	5·31	968278	·83	598354	6·14	401646	22
39	566951	5·30	968228	·84	598722	6·14	401278	21
40	567269	5·30	968178	·84	599091	6·13	400909	20
41	9·567587	5·29	9·968128	·84	9·599459	6·13	10·400541	19
42	567904	5·29	968078	·84	599827	6·13	400173	18
43	568222	5·28	968027	·84	600194	6·12	399806	17
44	568539	5·28	967977	·84	600562	6·12	399438	16
45	568856	5·28	967927	·84	600929	6·11	399071	15
46	569172	5·27	967876	·84	601296	6·11	398704	14
47	569488	5·27	967826	·84	601662	6·11	398338	13
48	569804	5·26	967775	·84	602029	6·10	397971	12
49	570120	5·26	967725	·84	602395	6·10	397605	11
50	570435	5·25	967674	·84	602761	6·10	397239	10
51	9·570751	5·25	9·967624	·84	9·603127	6·09	10·396873	9
52	571066	5·24	967573	·84	603493	6·09	396507	8
53	571380	5·24	967522	·85	603858	6·09	396142	7
54	571695	5·23	967471	·85	604223	6·08	395777	6
55	572009	5·23	967421	·85	604588	6·08	395412	5
56	572323	5·23	967370	·85	604953	6·07	395047	4
57	572636	5·22	967319	·85	605317	6·07	394683	3
58	572950	5·22	967268	·85	605682	6·07	394318	2
59	573263	5·21	967217	·85	606046	6·06	393954	1
60	573575	5·21	967166	·85	606410	6·06	393590	0
	Cosine	D.	Sine	68°	Cotang.	D.	Tang.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9·573575	5·21	9·967166	·85	9·606410	6·06	10·393590	60
1	573888	5·20	967115	·85	606773	6·06	393227	59
2	574200	5·20	967064	·85	607137	6·05	392863	58
3	574512	5·19	967013	·85	607500	6·05	392500	57
4	574824	5·19	966961	·85	607863	6·04	392137	56
5	575136	5·19	966910	·85	608225	6·04	391775	55
6	575447	5·18	966859	·85	608588	6·04	391412	54
7	575758	5·18	966808	·85	608950	6·03	391050	53
8	576069	5·17	966756	·86	609312	6·03	390688	52
9	576379	5·17	966705	·86	609674	6·03	390326	51
10	576689	5·16	966653	·86	610036	6·02	389964	50
11	9·576999	5·16	9·966602	·86	9·610397	6·02	10·389603	49
12	577309	5·16	966550	·86	610759	6·02	389241	48
13	577618	5·15	966499	·86	611120	6·01	388880	47
14	577927	5·15	966447	·86	611480	6·01	388520	46
15	578236	5·14	966395	·86	611841	6·01	388159	45
16	578545	5·14	966344	·86	612201	6·00	387799	44
17	578853	5·13	966292	·86	612561	6·00	387439	43
18	579162	5·13	966240	·86	612921	6·00	387079	42
19	579470	5·13	966188	·86	613281	5·99	386719	41
20	579777	5·12	966136	·86	613641	5·99	386359	40
21	9·580085	5·12	9·966085	·87	9·614000	5·98	10·386000	39
22	580392	5·11	966033	·87	614359	5·98	385641	38
23	580699	5·11	965981	·87	614718	5·98	385282	37
24	581005	5·11	965928	·87	615077	5·97	384923	36
25	581312	5·10	965876	·87	615435	5·97	384565	35
26	581618	5·10	965824	·87	615793	5·97	384207	34
27	581924	5·09	965772	·87	616151	5·96	383849	33
28	582229	5·09	965720	·87	616509	5·96	383491	32
29	582535	5·09	965668	·87	616867	5·96	383133	31
30	582840	5·08	965615	·87	617224	5·95	382776	30
31	9·583145	5·08	9·965563	·87	9·617582	5·95	10·382418	29
32	583449	5·07	965511	·87	617939	5·95	382061	28
33	583754	5·07	965458	·87	618295	5·94	381705	27
34	584058	5·06	965406	·87	618652	5·94	381348	26
35	584361	5·06	965353	·88	619008	5·94	380992	25
36	584665	5·06	965301	·88	619364	5·93	380636	24
37	584968	5·05	965248	·88	619721	5·93	380279	23
38	585272	5·05	965195	·88	620076	5·93	379924	22
39	585574	5·04	965143	·88	620432	5·92	379568	21
40	585877	5·04	965090	·88	620787	5·92	379213	20
41	9·586179	5·03	9·965037	·88	9·621142	5·92	10·378858	19
42	586482	5·03	964984	·88	621497	5·91	378503	18
43	586783	5·03	964931	·88	621852	5·91	378148	17
44	587085	5·02	964879	·88	622207	5·90	377793	16
45	587386	5·02	964826	·88	622561	5·90	377439	15
46	587688	5·01	964773	·88	622915	5·90	377085	14
47	587989	5·01	964719	·88	623269	5·89	376731	13
48	588289	5·01	964666	·89	623623	5·89	376377	12
49	588590	5·00	964613	·89	623976	5·89	376024	11
50	588890	5·00	964560	·89	624330	5·88	375670	10
51	9·589190	4·99	9·964507	·89	9·624683	5·88	10·375317	9
52	589489	4·99	964454	·89	625036	5·88	374964	8
53	589789	4·99	964400	·89	625388	5·87	374612	7
54	590088	4·98	964347	·89	625741	5·87	374259	6
55	590387	4·98	964294	·89	626093	5·87	373907	5
56	590686	4·97	964240	·89	626445	5·86	373555	4
57	590984	4·97	964187	·89	626797	5·86	373203	3
58	591282	4·97	964133	·89	627149	5·86	372851	2
59	591580	4·96	964080	·89	627501	5·85	372499	1
60	591878	4·96	964026	·89	627852	5·85	372148	0
	Cosine	D.	Sine	67°	Cotang.	D.	Tang.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9·591878	4·96	9·964026	·89	9·627852	5·85	10·372148	60
1	592176	4·95	963972	·89	628203	5·85	371797	59
2	592473	4·95	963919	·89	628554	5·85	371446	58
3	592770	4·95	963865	·90	628905	5·84	371095	57
4	593067	4·94	963811	·90	629255	5·84	370745	56
5	593363	4·94	963757	·90	629606	5·83	370394	55
6	593659	4·93	963704	·90	629956	5·83	370044	54
7	593955	4·93	963650	·90	630306	5·83	369694	53
8	594251	4·93	963596	·90	630656	5·83	369344	52
9	594547	4·92	963542	·90	631005	5·82	368995	51
10	594842	4·92	963488	·90	631355	5·82	368645	50
11	9·595137	4·91	9·963434	·90	9·631704	5·82	10·368296	49
12	595432	4·91	963379	·90	632053	5·81	367947	48
13	595727	4·91	963325	·90	632401	5·81	367599	47
14	596021	4·90	963271	·90	632750	5·81	367250	46
15	596315	4·90	963217	·90	633098	5·80	366902	45
16	596609	4·89	963163	·90	633447	5·80	366553	44
17	596903	4·89	963108	·91	633795	5·80	366205	43
18	597196	4·89	963054	·91	634143	5·79	365857	42
19	597490	4·88	962999	·91	634490	5·79	365510	41
20	597783	4·88	962945	·91	634838	5·79	365162	40
21	9·598075	4·87	9·962890	·91	9·635185	5·78	10·364815	39
22	598368	4·87	962836	·91	635532	5·78	364468	38
23	598660	4·87	962781	·91	635879	5·78	364121	37
24	598952	4·86	962727	·91	636226	5·77	363774	36
25	599244	4·86	962672	·91	636572	5·77	363428	35
26	599536	4·85	962617	·91	636919	5·77	363081	34
27	599827	4·85	962562	·91	637265	5·77	362735	33
28	600118	4·85	962508	·91	637611	5·76	362389	32
29	600409	4·84	962453	·91	637956	5·76	362044	31
30	600700	4·84	962398	·92	638302	5·76	361698	30
31	9·600990	4·84	9·962343	·92	9·638647	5·75	10·361353	29
32	601280	4·83	962288	·92	638992	5·75	361008	28
33	601570	4·83	962233	·92	639337	5·75	360663	27
34	601860	4·82	962178	·92	639682	5·74	360318	26
35	602150	4·82	962123	·92	640027	5·74	359973	25
36	602439	4·82	962067	·92	640371	5·74	359629	24
37	602728	4·81	962012	·92	640716	5·73	359284	23
38	603017	4·81	961957	·92	641060	5·73	358940	22
39	603305	4·81	961902	·92	641404	5·73	358596	21
40	603594	4·80	961846	·92	641747	5·72	358253	20
41	9·603882	4·80	9·961791	·92	9·642091	5·72	10·357909	19
42	604170	4·79	961735	·92	642434	5·72	357566	18
43	604457	4·79	961680	·92	642777	5·72	357223	17
44	604745	4·79	961624	·93	643120	5·71	356880	16
45	605032	4·78	961569	·93	643463	5·71	356537	15
46	605319	4·78	961513	·93	643806	5·71	356194	14
47	605606	4·78	961458	·93	644148	5·70	355852	13
48	605892	4·77	961402	·93	644490	5·70	355510	12
49	606179	4·77	961346	·93	644832	5·70	355168	11
50	606465	4·76	961290	·93	645174	5·69	354826	10
51	9·606751	4·76	9·961235	·93	9·645516	5·69	10·354484	9
52	607036	4·76	961179	·93	645857	5·69	354143	8
53	607322	4·75	961123	·93	646199	5·69	353801	7
54	607607	4·75	961067	·93	646540	5·68	353460	6
55	607892	4·74	961011	·93	646881	5·68	353119	5
56	608177	4·74	960955	·93	647222	5·68	352778	4
57	608461	4·74	960899	·93	647562	5·67	352438	3
58	608745	4·73	960843	·94	647903	5·67	352097	2
59	609029	4·73	960786	·94	648243	5·67	351757	1
60	609313	4·73	960730	·94	648583	5·66	351417	0
	Cosine	D.	Sine	66°	Cotang.	D.	Tang.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9·609313	4·73	9·960730	·94	9·648583	5·66	10·351417	60
1	609597	4·72	960674	·94	648923	5·66	351077	59
2	609880	4·72	960618	·94	649263	5·66	350737	58
3	610164	4·72	960561	·94	649602	5·66	350398	57
4	610447	4·71	960505	·94	649942	5·65	350058	56
5	610729	4·71	960448	·94	650281	5·65	349719	55
6	611012	4·70	960392	·94	650620	5·65	349380	54
7	611294	4·70	960335	·94	650959	5·64	349041	53
8	611576	4·70	960279	·94	651297	5·64	348703	52
9	611858	4·69	960222	·94	651636	5·64	348364	51
10	612140	4·69	960165	·94	651974	5·63	348026	50
11	9·612421	4·69	9·960109	·95	9·652312	5·63	10·347688	49
12	612702	4·68	960052	·95	652650	5·63	347350	48
13	612983	4·68	959995	·95	652988	5·63	347012	47
14	613264	4·67	959938	·95	653326	5·62	346674	46
15	613545	4·67	959882	·95	653663	5·62	346337	45
16	613825	4·67	959825	·95	654000	5·62	346000	44
17	614105	4·66	959768	·95	654337	5·61	345663	43
18	614385	4·66	959711	·95	654674	5·61	345326	42
19	614665	4·66	959654	·95	655011	5·61	344989	41
20	614944	4·65	959596	·95	655348	5·61	344652	40
21	9·615223	4·65	9·959539	·95	9·655684	5·60	10·344316	39
22	615502	4·65	959482	·95	656020	5·60	343980	38
23	615781	4·64	959425	·95	656356	5·60	343644	37
24	616060	4·64	959368	·95	656692	5·59	343308	36
25	616338	4·64	959310	·96	657028	5·59	342972	35
26	616616	4·63	959253	·96	657364	5·59	342636	34
27	616894	4·63	959195	·96	657699	5·59	342301	33
28	617172	4·62	959138	·96	658034	5·58	341966	32
29	617450	4·62	959081	·96	658369	5·58	341631	31
30	617727	4·62	959023	·96	658704	5·58	341296	30
31	9·618004	4·61	9·958965	·96	9·659039	5·58	10·340961	29
32	618281	4·61	958908	·96	659373	5·57	340627	28
33	618558	4·61	958850	·96	659708	5·57	340292	27
34	618834	4·60	958792	·96	660042	5·57	339958	26
35	619110	4·60	958734	·96	660376	5·57	339624	25
36	619386	4·60	958677	·96	660710	5·56	339290	24
37	619662	4·59	958619	·96	661043	5·56	338957	23
38	619938	4·59	958561	·96	661377	5·56	338623	22
39	620213	4·59	958503	·97	661710	5·55	338290	21
40	620488	4·58	958445	·97	662043	5·55	337957	20
41	9·620763	4·58	9·958387	·97	9·662376	5·55	10·337624	19
42	621038	4·57	958329	·97	662709	5·54	337291	18
43	621313	4·57	958271	·97	663042	5·54	336958	17
44	621587	4·57	958213	·97	663375	5·54	336625	16
45	621861	4·56	958154	·97	663707	5·54	336293	15
46	622135	4·56	958096	·97	664039	5·53	335961	14
47	622409	4·56	958038	·97	664371	5·53	335629	13
48	622682	4·55	957979	·97	664703	5·53	335297	12
49	622956	4·55	957921	·97	665035	5·53	334965	11
50	623229	4·55	957863	·97	665366	5·52	334634	10
51	9·623502	4·54	9·957804	·97	9·665697	5·52	10·334303	9
52	623774	4·54	957746	·98	666029	5·52	333971	8
53	624047	4·54	957687	·98	666360	5·51	333640	7
54	624319	4·53	957628	·98	666691	5·51	333309	6
55	624591	4·53	957570	·98	667021	5·51	332979	5
56	624863	4·53	957511	·98	667352	5·51	332648	4
57	625135	4·52	957452	·98	667682	5·50	332318	3
58	625406	4·52	957393	·98	668013	5·50	331987	2
59	625677	4·52	957335	·98	668343	5·50	331657	1
60	625948	4·51	957276	·98	668672	5·50	331328	0
	Cosine	D.	Sine	65°	Cotang.	D.	Tang.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9·625948	4·51	9·957276	·98	9·668673	5·50	10·331327	60
1	626219	4·51	957217	·98	669002	5·49	330998	59
2	626490	4·51	957158	·98	669332	5·49	330668	58
3	626760	4·50	957099	·98	669661	5·49	330339	57
4	627030	4·50	957040	·98	669991	5·48	330009	56
5	627300	4·50	956981	·98	670320	5·48	329680	55
6	627570	4·49	956921	·99	670649	5·48	329351	54
7	627840	4·49	956862	·99	670977	5·48	329023	53
8	628109	4·49	956803	·99	671306	5·47	328694	52
9	628378	4·48	956744	·99	671634	5·47	328366	51
10	628647	4·48	956684	·99	671963	5·47	328037	50
11	9·628916	4·47	9·956625	·99	9·672291	5·47	10·327709	49
12	629185	4·47	956566	·99	672619	5·46	327381	48
13	629453	4·47	956506	·99	672947	5·46	327053	47
14	629721	4·46	956447	·99	673274	5·46	326726	46
15	629989	4·46	956387	·99	673602	5·46	326398	45
16	630257	4·46	956327	·99	673929	5·45	326071	44
17	630524	4·46	956268	·99	674257	5·45	325743	43
18	630792	4·45	956208	1·00	674584	5·45	325416	42
19	631059	4·45	956148	1·00	674910	5·44	325090	41
20	631326	4·45	956089	1·00	675237	5·44	324763	40
21	9·631593	4·44	9·956029	1·00	9·675564	5·44	10·324436	39
22	631859	4·44	955969	1·00	675890	5·44	324110	38
23	632125	4·44	955909	1·00	676216	5·43	323784	37
24	632392	4·43	955849	1·00	676543	5·43	323457	36
25	632658	4·43	955789	1·00	676869	5·43	323131	35
26	632923	4·43	955729	1·00	677194	5·43	322806	34
27	633189	4·42	955669	1·00	677520	5·42	322480	33
28	633454	4·42	955609	1·00	677846	5·42	322154	32
29	633719	4·42	955548	1·00	678171	5·42	321829	31
30	633984	4·41	955488	1·00	678496	5·42	321504	30
31	9·634249	4·41	9·955428	1·01	9·678821	5·41	10·321179	29
32	634514	4·40	955368	1·01	679146	5·41	320854	28
33	634778	4·40	955307	1·01	679471	5·41	320529	27
34	635042	4·40	955247	1·01	679795	5·41	320205	26
35	635306	4·39	955186	1·01	680120	5·40	319880	25
36	635570	4·39	955126	1·01	680444	5·40	319556	24
37	635834	4·39	955065	1·01	680768	5·40	319232	23
38	636097	4·38	955005	1·01	681092	5·40	318908	22
39	636360	4·38	954944	1·01	681416	5·39	318584	21
40	636623	4·38	954883	1·01	681740	5·39	318260	20
41	9·636886	4·37	9·954823	1·01	9·682063	5·39	10·317937	19
42	637148	4·37	954762	1·01	682387	5·39	317613	18
43	637411	4·37	954701	1·01	682710	5·38	317290	17
44	637673	4·37	954640	1·01	683033	5·38	316967	16
45	637935	4·36	954579	1·01	683356	5·38	316644	15
46	638197	4·36	954518	1·02	683679	5·38	316321	14
47	638458	4·36	954457	1·02	684001	5·37	315999	13
48	638720	4·35	954396	1·02	684324	5·37	315676	12
49	638981	4·35	954335	1·02	684646	5·37	315354	11
50	639242	4·35	954274	1·02	684968	5·37	315032	10
51	9·639503	4·34	9·954213	1·02	9·685290	5·36	10·314710	9
52	639764	4·34	954152	1·02	685612	5·36	314388	8
53	640024	4·34	954090	1·02	685934	5·36	314066	7
54	640284	4·33	954029	1·02	686255	5·36	313745	6
55	640544	4·33	953968	1·02	686577	5·35	313423	5
56	640804	4·33	953906	1·02	686898	5·35	313102	4
57	641064	4·32	953845	1·02	687219	5·35	312781	3
58	641324	4·32	953783	1·02	687540	5·35	312460	2
59	641584	4·32	953722	1·03	687861	5·34	312139	1
60	641842	4·31	953660	1·03	688182	5·34	311818	0
	Cosine	D.	Sine	64°	Cotang.	D.	Tang.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9·641842	4·31	9·953660	1·03	9·688182	5·34	10·311818	60
1	642101	4·31	953599	1·03	688502	5·34	311498	59
2	642360	4·31	953537	1·03	688823	5·34	311177	58
3	642618	4·30	953475	1·03	689143	5·33	310857	57
4	642877	4·30	953413	1·03	689463	5·33	310537	56
5	643135	4·30	953352	1·03	689783	5·33	310217	55
6	643393	4·30	953290	1·03	690103	5·33	309897	54
7	643650	4·29	953228	1·03	690423	5·33	309577	53
8	643908	4·29	953166	1·03	690742	5·32	309258	52
9	644165	4·29	953104	1·03	691062	5·32	308938	51
10	644423	4·28	953042	1·03	691381	5·32	308619	50
11	9·644680	4·28	9·952980	1·04	9·691700	5·31	10·308300	49
12	644936	4·28	952918	1·04	692019	5·31	307981	48
13	645193	4·27	952855	1·04	692338	5·31	307662	47
14	645450	4·27	952793	1·04	692656	5·31	307344	46
15	645706	4·27	952731	1·04	692975	5·31	307025	45
16	645962	4·26	952669	1·04	693293	5·30	306707	44
17	646218	4·26	952606	1·04	693612	5·30	306388	43
18	646474	4·26	952544	1·04	693930	5·30	306070	42
19	646729	4·25	952481	1·04	694248	5·30	305752	41
20	646984	4·25	952419	1·04	694566	5·29	305434	40
21	9·647240	4·25	9·952356	1·04	9·694883	5·29	10·305117	39
22	647494	4·24	952294	1·04	695201	5·29	304799	38
23	647749	4·24	952231	1·04	695518	5·29	304482	37
24	648004	4·24	952168	1·05	695836	5·29	304164	36
25	648258	4·24	952106	1·05	696153	5·28	303847	35
26	648512	4·23	952043	1·05	696470	5·28	303530	34
27	648766	4·23	951980	1·05	696787	5·28	303213	33
28	649020	4·23	951917	1·05	697103	5·28	302897	32
29	649274	4·22	951854	1·05	697420	5·27	302580	31
30	649527	4·22	951791	1·05	697736	5·27	302264	30
31	9·649781	4·22	9·951728	1·05	9·698053	5·27	10·301947	29
32	650034	4·22	951665	1·05	698369	5·27	301631	28
33	650287	4·21	951602	1·05	698685	5·26	301315	27
34	650539	4·21	951539	1·05	699001	5·26	300999	26
35	650792	4·21	951476	1·05	699316	5·26	300684	25
36	651044	4·20	951412	1·05	699632	5·26	300368	24
37	651297	4·20	951349	1·06	699947	5·26	300053	23
38	651549	4·20	951286	1·06	700263	5·25	299737	22
39	651800	4·19	951222	1·06	700578	5·25	299422	21
40	652052	4·19	951159	1·06	700893	5·25	299107	20
41	9·652304	4·19	9·951096	1·06	9·701208	5·24	10·298792	19
42	652555	4·18	951032	1·06	701523	5·24	298477	18
43	652806	4·18	950968	1·06	701837	5·24	298163	17
44	653057	4·18	950905	1·06	702152	5·24	297848	16
45	653308	4·18	950841	1·06	702466	5·24	297534	15
46	653558	4·17	950778	1·06	702780	5·23	297220	14
47	653808	4·17	950714	1·06	703095	5·23	296905	13
48	654059	4·17	950650	1·06	703409	5·23	296591	12
49	654309	4·16	950586	1·06	703723	5·23	296277	11
50	654558	4·16	950522	1·07	704036	5·22	295964	10
51	9·654808	4·16	9·950458	1·07	9·704350	5·22	10·295650	9
52	655058	4·16	950394	1·07	704663	5·22	295337	8
53	655307	4·15	950330	1·07	704977	5·22	295023	7
54	655556	4·15	950266	1·07	705290	5·22	294710	6
55	655805	4·15	950202	1·07	705603	5·21	294397	5
56	656054	4·14	950138	1·07	705916	5·21	294084	4
57	656302	4·14	950074	1·07	706228	5·21	293772	3
58	656551	4·14	950010	1·07	706541	5·21	293459	2
59	656799	4·13	949945	1·07	706854	5·21	293146	1
60	657047	4·13	949881	1·07	707166	5·20	292834	0
	Cosine	D.	Sine	63°	Cotang.	D.	Tang.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9·657047	4·13	9·949881	1·07	9·707166	5·20	10·292834	60
1	657295	4·13	949816	1·07	707478	5·20	292522	59
2	657542	4·12	949752	1·07	707790	5·20	292210	58
3	657790	4·12	949688	1·08	708102	5·20	291898	57
4	658037	4·12	949623	1·08	708414	5·19	291586	56
5	658284	4·12	949558	1·08	708726	5·19	291274	55
6	658531	4·11	949494	1·08	709037	5·19	290963	54
7	658778	4·11	949429	1·08	709349	5·19	290651	53
8	659025	4·11	949364	1·08	709660	5·19	290340	52
9	659271	4·10	949300	1·08	709971	5·18	290029	51
10	659517	4·10	949235	1·08	710282	5·18	289718	50
11	9·659763	4·10	9·949170	1·08	9·710593	5·18	10·289407	49
12	660009	4·09	949105	1·08	710904	5·18	289096	48
13	660255	4·09	949040	1·08	711215	5·18	288785	47
14	660501	4·09	948975	1·08	711525	5·17	288475	46
15	660746	4·09	948910	1·08	711836	5·17	288164	45
16	660991	4·08	948845	1·08	712146	5·17	287854	44
17	661236	4·08	948780	1·09	712456	5·17	287544	43
18	661481	4·08	948715	1·09	712766	5·16	287234	42
19	661726	4·07	948650	1·09	713076	5·16	286924	41
20	661970	4·07	948584	1·09	713386	5·16	286614	40
21	9·662214	4·07	9·948519	1·09	9·713696	5·16	10·286304	39
22	662459	4·07	948454	1·09	714005	5·16	285995	38
23	662703	4·06	948388	1·09	714314	5·15	285686	37
24	662946	4·06	948323	1·09	714624	5·15	285376	36
25	663190	4·06	948257	1·09	714933	5·15	285067	35
26	663433	4·05	948192	1·09	715242	5·15	284758	34
27	663677	4·05	948126	1·09	715551	5·14	284449	33
28	663920	4·05	948060	1·09	715860	5·14	284140	32
29	664163	4·05	947995	1·10	716168	5·14	283832	31
30	664406	4·04	947929	1·10	716477	5·14	283523	30
31	9·664648	4·04	9·947863	1·10	9·716785	5·14	10·283215	29
32	664891	4·04	947797	1·10	717093	5·13	282907	28
33	665133	4·03	947731	1·10	717401	5·13	282599	27
34	665375	4·03	947665	1·10	717709	5·13	282291	26
35	665617	4·03	947600	1·10	718017	5·13	281983	25
36	665859	4·02	947533	1·10	718325	5·13	281670	24
37	666100	4·02	947467	1·10	718633	5·12	281367	23
38	666342	4·02	947401	1·10	718940	5·12	281060	22
39	666583	4·02	947335	1·10	719248	5·12	280752	21
40	666824	4·01	947269	1·10	719555	5·12	280445	20
41	9·667065	4·01	9·947203	1·10	9·719862	5·12	10·280138	19
42	667305	4·01	947136	1·11	720169	5·11	279831	18
43	667546	4·01	947070	1·11	720476	5·11	279524	17
44	667786	4·00	947004	1·11	720783	5·11	279217	16
45	668027	4·00	946937	1·11	721089	5·11	278911	15
46	668267	4·00	946871	1·11	721396	5·11	278604	14
47	668506	3·99	946804	1·11	721702	5·10	278298	13
48	668746	3·99	946738	1·11	722009	5·10	277991	12
49	668986	3·99	946671	1·11	722315	5·10	277685	11
50	669225	3·99	946604	1·11	722621	5·10	277379	10
51	9·669464	3·98	9·946538	1·11	9·722927	5·10	10·277073	9
52	669703	3·98	946471	1·11	723232	5·09	276768	8
53	669942	3·98	946404	1·11	723538	5·09	276462	7
54	670181	3·97	946337	1·11	723844	5·09	276156	6
55	670419	3·97	946270	1·12	724149	5·09	275851	5
56	670658	3·97	946203	1·12	724454	5·09	275546	4
57	670896	3·97	946136	1·12	724759	5·08	275241	3
58	671134	3·96	946069	1·12	725065	5·08	274935	2
59	671372	3·96	946002	1·12	725369	5·08	274631	1
60	671609	3·96	945935	1·12	725674	5·08	274326	0
	Cosine	D.	Sine	62°	Cotang.	D.	Tang.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9·671609	3·96	9·945935	1·12	9·725674	5·08	10·274326	60
1	671847	3·95	945868	1·12	725979	5·08	274021	59
2	672084	3·95	945800	1·12	726284	5·07	273716	58
3	672321	3·95	945733	1·12	726588	5·07	273412	57
4	672558	3·95	945666	1·12	726892	5·07	273108	56
5	672795	3·94	945598	1·12	727197	5·07	272803	55
6	673032	3·94	945531	1·12	727501	5·07	272499	54
7	673268	3·94	945464	1·13	727805	5·06	272195	53
8	673505	3·94	945396	1·13	728109	5·06	271891	52
9	673741	3·93	945328	1·13	728412	5·06	271588	51
10	673977	3·93	945261	1·13	728716	5·06	271284	50
11	9·674213	3·93	9·945193	1·13	9·729020	5·06	10·270980	49
12	674448	3·92	945125	1·13	729323	5·05	270677	48
13	674684	3·92	945058	1·13	729626	5·05	270374	47
14	674919	3·92	944990	1·13	729929	5·05	270071	46
15	675155	3·92	944922	1·13	730233	5·05	269767	45
16	675390	3·91	944854	1·13	730535	5·05	269465	44
17	675624	3·91	944786	1·13	730838	5·04	269162	43
18	675859	3·91	944718	1·13	731141	5·04	268859	42
19	676094	3·91	944650	1·13	731444	5·04	268556	41
20	676328	3·90	944582	1·14	731746	5·04	268254	40
21	9·676562	3 90	9·944514	1·14	9·732048	5·04	10·267952	39
22	676796	3·90	944446	1·14	732351	5·03	267649	38
23	677030	3·90	944377	1·14	732653	5·03	267347	37
24	677264	3·89	944309	1·14	732955	5·03	267045	36
25	677498	3·89	944241	1·14	733257	5·03	266743	35
26	677731	3·89	944172	1·14	733558	5·03	266442	34
27	677964	3·88	944104	1·14	733860	5·02	266140	33
28	678197	3·88	944036	1·14	734162	5·02	265838	32
29	678430	3·88	943967	1·14	734463	5·02	265537	31
30	678663	3·88	943899	1·14	734764	5·02	265236	30
31	9·678895	3·87	9·943830	1·14	9·735066	5·02	10·264934	29
32	679128	3·87	943761	1·14	735367	5·02	264633	28
33	679360	3·87	943693	1·15	735668	5·01	264332	27
34	679592	3·87	943624	1·15	735969	5·01	264031	26
35	679824	3·86	943555	1·15	736269	5·01	263731	25
36	680056	3·86	943486	1·15	736570	5·01	263430	24
37	680288	3·86	943417	1·15	736871	5·01	263129	23
38	680519	3·85	943348	1·15	737171	5·00	262829	22
39	680750	3·85	943279	1·15	737471	5·00	262529	21
40	680982	3·85	943210	1·15	737771	5·00	262229	20
41	9·681213	3·85	9·943141	1·15	9·738071	5·00	10·261929	19
42	681443	3·84	943072	1·15	738371	5·00	261629	18
43	681674	3·84	943003	1·15	738671	4·99	261329	17
44	681905	3·84	942934	1·15	738971	4·99	261029	16
45	682135	3·84	942864	1·15	739271	4·99	260729	15
46	682365	3·83	942795	1·16	739570	4·99	260430	14
47	682595	3·83	942726	1·16	739870	4·99	260130	13
48	682825	3·83	942656	1·16	740169	4·99	259831	12
49	683055	3·83	942587	1·16	740468	4·98	259532	11
50	683284	3·82	942517	1·16	740767	4·98	259233	10
51	9·683514	3·82	9·942448	1·16	9·741066	4·98	10·258934	9
52	683743	3·82	942378	1·16	741365	4·98	258635	8
53	683972	3·82	942308	1·16	741664	4·98	258336	7
54	684201	3·81	942239	1·16	741962	4·97	258038	6
55	684430	3·81	942169	1·16	742261	4·97	257739	5
56	684658	3·81	942099	1·16	742559	4·97	257441	4
57	684887	3·80	942029	1·16	742858	4·97	257142	3
58	685115	3·80	941959	1·16	743156	4·97	256844	2
59	685343	3·80	941889	1·17	743454	4·97	256546	1
60	685571	3·80	941819	1·17	743752	4·96	256248	0
	Cosine	D.	Sine	61°	Cotang.	D.	Tang.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9·685571	3·80	9·941819	1·17	9·743752	4·96	10·256248	60
1	685799	3·79	941749	1·17	744050	4·96	255950	59
2	686027	3·79	941679	1·17	744348	4·96	255652	58
3	686254	3·79	941609	1·17	744645	4·96	255355	57
4	686482	3·79	941539	1·17	744943	4·96	255057	56
5	686709	3·78	941469	1·17	745240	4·96	254760	55
6	686936	3·78	941398	1·17	745538	4·95	254462	54
7	687163	3·78	941328	1·17	745835	4·95	254165	53
8	687389	3·78	941258	1·17	746132	4·95	253868	52
9	687616	3·77	941187	1·17	746429	4·95	253571	51
10	687843	3·77	941117	1·17	746726	4·95	253274	50
11	9·688069	3·77	9·941046	1·18	9·747023	4·94	10·252977	49
12	688295	3·77	940975	1·18	747319	4·94	252681	48
13	688521	3·76	940905	1·18	747616	4·94	252384	47
14	688747	3·76	940834	1·18	747913	4·94	252087	46
15	688972	3·76	940763	1·18	748209	4·94	251791	45
16	689198	3·76	940693	1·18	748505	4·93	251495	44
17	689423	3·75	940622	1·18	748801	4·93	251199	43
18	689648	3·75	940551	1·18	749097	4·93	250903	42
19	689873	3·75	940480	1·18	749393	4·93	250607	41
20	690098	3·75	940409	1·18	749689	4·93	250311	40
21	9·690323	3·74	9·940338	1·18	9·749985	4·93	10·250015	39
22	690548	3·74	940267	1·18	750281	4·92	249719	38
23	690772	3·74	940196	1·18	750576	4·92	249424	37
24	690996	3·74	940125	1·19	750872	4·92	249128	36
25	691220	3·73	940054	1·19	751167	4·92	248833	35
26	691444	3·73	939982	1·19	751462	4·92	248538	34
27	691668	3·73	939911	1·19	751757	4·92	248243	33
28	691892	3·73	939840	1·19	752052	4·91	247948	32
29	692115	3·72	939768	1·19	752347	4·91	247653	31
30	692339	3·72	939697	1·19	752642	4·91	247358	30
31	9·692562	3·72	9·939625	1·19	9·752937	4·91	10·247063	29
32	692785	3·71	939554	1·19	753231	4·91	246769	28
33	693008	3·71	939482	1·19	753526	4·91	246474	27
34	693231	3·71	939410	1·19	753820	4·90	246180	26
35	693453	3·71	939339	1·19	754115	4·90	245885	25
36	693676	3·70	939267	1·20	754409	4·90	245591	24
37	693898	3·70	939195	1·20	754703	4·90	245297	23
38	694120	3·70	939123	1·20	754997	4·90	245003	22
39	694342	3·70	939052	1·20	755291	4·90	244709	21
40	694564	3·69	938980	1·20	755585	4·89	244415	20
41	9·694786	3·69	9·938908	1·20	9·755878	4·89	10·244122	19
42	695007	3·69	938836	1·20	756172	4·89	243828	18
43	695229	3·69	938763	1·20	756465	4·89	243535	17
44	695450	3·68	938691	1·20	756759	4·89	243241	16
45	695671	3·68	938619	1·20	757052	4·89	242948	15
46	695892	3·68	938547	1·20	757345	4·88	242655	14
47	696113	3·68	938475	1·20	757638	4·88	242362	13
48	696334	3·67	938402	1·21	757931	4·88	242069	12
49	696554	3·67	938330	1·21	758224	4·88	241776	11
50	696775	3·67	938258	1·21	758517	4·88	241483	10
51	9·696995	3·67	9·938185	1·21	9·758810	4·88	10·241190	9
52	697215	3·66	938113	1·21	759102	4·87	240898	8
53	697435	3·66	938040	1·21	759395	4·87	240605	7
54	697654	3·66	937967	1·21	759687	4·87	240313	6
55	697874	3·66	937895	1·21	759979	4·87	240021	5
56	698094	3·65	937822	1·21	760272	4·87	239728	4
57	698313	3·65	937749	1·21	760564	4·87	239436	3
58	698532	3·65	937676	1·21	760856	4·86	239144	2
59	698751	3·65	937604	1·21	761148	4·86	238852	1
60	698970	3·64	937531	1·21	761439	4·86	238561	0
	Cosine	D.	Sine	60°	Cotang.	D.	Tang.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9·698970	3·64	9·937531	1·21	9·761439	4·86	10·238561	60
1	699189	3·64	937458	1·22	761731	4·86	238269	59
2	699407	3·64	937385	1·22	762023	4·86	237977	58
3	699626	3·64	937312	1·22	762314	4·86	237686	57
4	699844	3·63	937238	1·22	762606	4·85	237394	56
5	700062	3·63	937165	1·22	762897	4·85	237103	55
6	700280	3·63	937092	1·22	763188	4·85	236812	54
7	700498	3·63	937019	1·22	763479	4·85	236521	53
8	700716	3·63	936946	1·22	763770	4·85	236230	52
9	700933	3·62	936872	1·22	764061	4·85	235939	51
10	701151	3·62	936799	1·22	764352	4·84	235648	50
11	9·701368	3·62	9·936725	1·22	9·764643	4·84	10·235357	49
12	701585	3·62	936652	1·23	764933	4·84	235067	48
13	701802	3·61	936578	1·23	765224	4·84	234776	47
14	702019	3·61	936505	1·23	765514	4·84	234486	46
15	702236	3·61	936431	1·23	765805	4·84	234195	45
16	702452	3·61	936357	1·23	766095	4·84	233905	44
17	702669	3·60	936284	1·23	766385	4·83	233615	43
18	702885	3·60	936210	1·23	766675	4·83	233325	42
19	703101	3·60	936136	1·23	766965	4·83	233035	41
20	703317	3·60	936062	1·23	767255	4·83	232745	40
21	9·703533	3·59	9·935988	1·23	9·767545	4·83	10·232455	39
22	703749	3·59	935914	1·23	767834	4·83	232166	38
23	703964	3·59	935840	1·23	768124	4·82	231876	37
24	704179	3·59	935766	1·24	768413	4·82	231587	36
25	704395	3·59	935692	1·24	768703	4·82	231297	35
26	704610	3·58	935618	1·24	768992	4·82	231008	34
27	704825	3·58	935543	1·24	769281	4·82	230719	33
28	705040	3·58	935469	1·24	769570	4·82	230430	32
29	705254	3·58	935395	1·24	769860	4·81	230140	31
30	705469	3·57	935320	1·24	770148	4·81	229852	30
31	9·705683	3·57	9·935246	1·24	9·770437	4·81	10·229563	29
32	705898	3·57	935171	1·24	770726	4·81	229274	28
33	706112	3·57	935097	1·24	771015	4·81	228985	27
34	706326	3·56	935022	1·24	771303	4·81	228697	26
35	706539	3·56	934948	1·24	771592	4·81	228408	25
36	706753	3·56	934873	1·24	771880	4·80	228120	24
37	706967	3·56	934798	1·25	772168	4·80	227832	23
38	707180	3·55	934723	1·25	772457	4·80	227543	22
39	707393	3·55	934649	1·25	772745	4·80	227255	21
40	707606	3·55	934574	1·25	773033	4·80	226967	20
41	9·707819	3·55	9·934499	1·25	9·773321	4·80	10·226679	19
42	708032	3·54	934424	1·25	773608	4·79	226392	18
43	708245	3·54	934349	1·25	773896	4·79	226104	17
44	708458	3·54	934274	1·25	774184	4·79	225816	16
45	708670	3·54	934199	1·25	774471	4·79	225529	15
46	708882	3·53	934123	1·25	774759	4·79	225241	14
47	709094	3·53	934048	1·25	775046	4·79	224954	13
48	709306	3·53	933973	1·25	775333	4·79	224667	12
49	709518	3·53	933898	1·26	775621	4·78	224379	11
50	709730	3·53	933822	1·26	775908	4·78	224092	10
51	9·709941	3·52	9·933747	1·26	9·776195	4·78	10·223805	9
52	710153	3·52	933671	1·26	776482	4·78	223518	8
53	710364	3·52	933596	1·26	776769	4·78	223231	7
54	710575	3·52	933520	1·26	777055	4·78	222945	6
55	710786	3·51	933445	1·26	777342	4·78	222658	5
56	710997	3·51	933369	1·26	777628	4·77	222372	4
57	711208	3·51	933293	1·26	777915	4·77	222085	3
58	711419	3·51	933217	1·26	778201	4·77	221799	2
59	711629	3·50	933141	1·26	778487	4·77	221512	1
60	711839	3·50	933066	1·26	778774	4·77	221226	0
	Cosine	D.	Sine	59°	Cotang.	D.	Tang.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9·711839	3·50	9·933066	1·26	9·778774	4·77	10·221226	60
1	712050	3·50	932990	1·27	779060	4·77	220940	59
2	712260	3·50	932914	1·27	779346	4·76	220654	58
3	712469	3·49	932838	1·27	779632	4·76	220368	57
4	712679	3·49	932762	1·27	779918	4·76	220082	56
5	712889	3·49	932685	1·27	780203	4·76	219797	55
6	713098	3·49	932609	1·27	780489	4·76	219511	54
7	713308	3·49	932533	1·27	780775	4·76	219225	53
8	713517	3·48	932457	1·27	781060	4·76	218940	52
9	713726	3·48	932380	1·27	781346	4·75	218654	51
10	713935	3·48	932304	1·27	781631	4·75	218369	50
11	9·714144	3·48	9·932228	1·27	9·781916	4·75	10·218084	49
12	714352	3·47	932151	1·27	782201	4·75	217799	48
13	14561	3·47	932075	1·28	782486	4·75	217514	47
14	714769	3·47	931998	1·28	782771	4·75	217229	46
15	714978	3·47	931921	1·28	783056	4·75	216944	45
16	715186	3·47	931845	1·28	783341	4·75	216659	44
17	715394	3·46	931768	1·28	783626	4·74	216374	43
18	715602	3·46	931691	1·28	783910	4·74	216090	42
19	715809	3·46	931614	1·28	784195	4·74	215805	41
20	716017	3·46	931537	1·28	784479	4·74	215521	40
21	9·716224	3·45	9·931460	1·28	9·784764	4·74	10·215236	39
22	716432	3·45	931383	1·28	785048	4·74	214952	38
23	716639	3·45	931306	1·28	785332	4·73	214668	37
24	716846	3·45	931229	1·29	785616	4·73	214384	36
25	717053	3·45	931152	1·29	785900	4·73	214100	35
26	717259	3·44	931075	1·29	786184	4·73	213816	34
27	717466	3·44	930998	1·29	786468	4·73	213532	33
28	717673	3·44	930921	1·29	786752	4·73	213248	32
29	717879	3·44	930843	1·29	787036	4·73	212964	31
30	718085	3·43	930766	1·29	787319	4·72	212681	30
31	9·718291	3·43	9·930688	1·29	9·787603	4·72	10·212397	29
32	718497	3·43	930611	1·29	787886	4·72	212114	28
33	718703	3·43	930533	1·29	788170	4·72	211830	27
34	718909	3·43	930456	1·29	788453	4·72	211547	26
35	719114	3·42	930378	1·29	788736	4·72	211264	25
36	719320	3·42	930300	1·30	789019	4·72	210981	24
37	719525	3·42	930223	1·30	789302	4·71	210698	23
38	719730	3·42	930145	1·30	789585	4·71	210415	22
39	719935	3·41	930067	1·30	789868	4·71	210132	21
40	720140	3·41	929989	1·30	790151	4·71	209849	20
41	9·720345	3·41	9·929911	1·30	9·790433	4·71	10·209567	19
42	720549	3·41	929833	1·30	790716	4·71	209284	18
43	720754	3·40	929755	1·30	790999	4·71	209001	17
44	720958	3·40	929677	1·30	791281	4·71	208719	16
45	721162	3·40	929599	1·30	791563	4·70	208437	15
46	721366	3·40	929521	1·30	791846	4·70	208154	14
47	721570	3·40	929442	1·30	792128	4·70	207872	13
48	721774	3·39	929364	1·31	792410	4·70	207590	12
49	721978	3·39	929286	1·31	792692	4·70	207308	11
50	722181	3·39	929207	1·31	792974	4·70	207026	10
51	9·722385	3·39	9·929129	1·31	9·793256	4·70	10·206744	9
52	722588	3·39	929050	1·31	793538	4·69	206462	8
53	722791	3·38	928972	1·31	793819	4·69	206181	7
54	722994	3·38	928893	1·31	794101	4·69	205899	6
55	723197	3·38	928815	1·31	794383	4·69	205617	5
56	723400	3·38	928736	1·31	794664	4·69	205336	4
57	723603	3·37	928657	1·31	794945	4·69	205055	3
58	723805	3·37	928578	1·31	795227	4·69	204773	2
59	724007	3·37	928499	1·31	795508	4·68	204492	1
60	724210	3·37	928420	1·31	795789	4·68	204211	0
	Cosine	D.	Sine	58°	Cotang.	D.	Tang.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9·724210	3·37	9·928420	1·32	9·795789	4·68	10·204211	60
1	724412	3·37	928342	1·32	796070	4·68	203930	59
2	724614	3·36	928263	1·32	796351	4·68	203649	58
3	724816	3·36	928183	1·32	796632	4·68	203368	57
4	725017	3·36	928104	1·32	796913	4·68	203087	56
5	725219	3·36	928025	1·32	797194	4·68	202806	55
6	725420	3·35	927946	1·32	797475	4·68	202525	54
7	725622	3·35	927867	1·32	797755	4·68	202245	53
8	725823	3·35	927787	1·32	798036	4·67	201964	52
9	726024	3·35	927708	1·32	798316	4·67	201684	51
10	726225	3·35	927629	1·32	798596	4·67	201404	50
11	9·726426	3·34	9·927549	1·32	9·798877	4·67	10·201123	49
12	726626	3·34	927470	1·33	799157	4·67	200843	48
13	726827	3·34	927390	1·33	799437	4·67	200563	47
14	727027	3·34	927310	1·33	799717	4·67	200283	46
15	727228	3·34	927231	1·33	799997	4·66	200003	45
16	727428	3·33	927151	1·33	800277	4·66	199723	44
17	727628	3·33	927071	1·33	800557	4·66	199443	43
18	727828	3·33	926991	1·33	800836	4·66	199164	42
19	728027	3·33	926911	1·33	801116	4·66	198884	41
20	728227	3·33	926831	1·33	801396	4·66	198604	40
21	9·728427	3·32	9·926751	1·33	9·801675	4·66	10·198325	39
22	728626	3·32	926671	1·33	801955	4·66	198045	38
23	728825	3·32	926591	1·33	802234	4·65	197766	37
24	729024	3·32	926511	1·34	802513	4·65	197487	36
25	729223	3·31	926431	1·34	802792	4·65	197208	35
26	729422	3·31	926351	1·34	803072	4·65	196928	34
27	729621	3·31	926270	1·34	803351	4·65	196649	33
28	729820	3·31	926190	1·34	803630	4·65	196370	32
29	730018	3·30	926110	1·34	803908	4·65	196092	31
30	730216	3·30	926029	1·34	804187	4·65	195813	30
31	9·730415	3·30	9·925949	1·34	9·804466	4·64	10·195534	29
32	730613	3·30	925868	1·34	804745	4·64	195255	28
33	730811	3·30	925788	1·34	805023	4·64	194977	27
34	731009	3·29	925707	1·34	805302	4·64	194698	26
35	731206	3·29	925626	1·34	805580	4·64	194420	25
36	731404	3·29	925545	1·35	805859	4·64	194141	24
37	731602	3·29	925465	1·35	806137	4·64	193863	23
38	731799	3·29	925384	1·35	806415	4·63	193585	22
39	731996	3·28	925303	1·35	806693	4·63	193307	21
40	732193	3·28	925222	1·35	806971	4·63	193029	20
41	9·732390	3·28	9·925141	1·35	9·807249	4·63	10·192751	19
42	732587	3·28	925060	1·35	807527	4·63	192473	18
43	732784	3·28	924979	1·35	807805	4·63	192195	17
44	732980	3·27	924897	1·35	808083	4·63	191917	16
45	733177	3·27	924816	1·35	808361	4·63	191639	15
46	733373	3·27	924735	1·36	808638	4·62	191362	14
47	733569	3·27	924654	1·36	808916	4·62	191084	13
48	733765	3·27	924572	1·36	809193	4·62	190807	12
49	733961	3·26	924491	1·36	809471	4·62	190529	11
50	734157	3·26	924409	1·36	809748	4·62	190252	10
51	9·734353	3·26	9·924328	1·36	9·810025	4·62	10·189975	9
52	734549	3·26	924246	1·36	810302	4·62	189698	8
53	734744	3·25	924164	1·36	810580	4·62	189420	7
54	734939	3·25	924083	1·36	810857	4·62	189143	6
55	735135	3·25	924001	1·36	811134	4·61	188866	5
56	735330	3·25	923919	1·36	811410	4·61	188590	4
57	735525	3·25	923837	1·36	811687	4·61	188313	3
58	735719	3·24	923755	1·37	811964	4·61	188036	2
59	735914	3·24	923673	1·37	812241	4·61	187759	1
60	736109	3·24	923591	1·37	812517	4·61	187483	0
	Cosine	D.	Sine	57°	Cotang.	D.	Tang.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9·736109	3·24	9·923591	1·37	9·812517	4·61	10·187483	60
1	736303	3·24	923509	1·37	812794	4·61	187206	59
2	736498	3·24	923427	1·37	813070	4·61	186930	58
3	736692	3·23	923345	1·37	813347	4·60	186653	57
4	736886	3·23	923263	1·37	813623	4·60	186377	56
5	737080	3·23	923181	1·37	813899	4·60	186101	55
6	737274	3·23	923098	1·37	814175	4·60	185825	54
7	737467	3·23	923016	1·37	814452	4·60	185548	53
8	737661	3·22	922933	1·37	814728	4·60	185272	52
9	737855	3·22	922851	1·37	815004	4·60	184996	51
10	738048	3·22	922768	1·38	815279	4·60	184721	50
11	9·738241	3·22	9·922686	1·38	9·815555	4·59	10·184445	49
12	738434	3·22	922603	1·38	815831	4·59	184169	48
13	738627	8·21	922520	1·38	816107	4·59	183893	47
14	738820	3·21	922438	1·38	816382	4·59	183618	46
15	739013	3·21	922355	1·38	816658	4·59	183342	45
16	739206	3·21	922272	1·38	816933	4·59	183067	44
17	739398	3·21	922189	1·38	817209	4·59	182791	43
18	739590	3·20	922106	1·38	817484	4·59	182516	42
19	739783	3·20	922023	1·38	817759	4·59	182241	41
20	739975	3·20	921940	1·38	818035	4·58	181965	40
21	9·740167	3·20	9·921857	1·39	9·818310	4·58	10·181690	39
22	740359	3·20	921774	1·39	818585	4·58	181415	38
23	740550	3·19	921691	1·39	818860	4·58	181140	37
24	740742	3·19	921607	1·39	819135	4·58	180865	36
25	740934	3·19	921524	1·39	819410	4·58	180590	35
26	741125	3·19	921441	1·39	819684	4·58	180316	34
27	741316	3·19	921357	1·39	819959	4·58	180041	33
28	741508	3·18	921274	1·39	820234	4·58	179766	32
29	741699	3·18	921190	1·39	820508	4·57	179492	31
30	741889	3·18	921107	1·39	820783	4·57	179217	30
31	9·742080	3·18	9·921023	1·39	9·821057	4·57	10·178943	29
32	742271	3·18	920939	1·40	821332	4·57	178668	28
33	742462	3·17	920856	1·40	821606	4·57	178394	27
34	742652	3·17	920772	1·40	821880	4·57	178120	26
35	742842	3·17	920688	1·40	822154	4·57	177846	25
36	743033	3·17	920604	1·40	822429	4·57	177571	24
37	743223	3·17	920520	1·40	822703	4·57	177297	23
38	743413	3·16	920436	1·40	822977	4·56	177023	22
39	743602	3·16	920352	1·40	823250	4·56	176750	21
40	743792	3·16	920268	1·40	823524	4·56	176476	20
41	9·743982	3·16	9·920184	1·40	9·823798	4·56	10·176202	19
42	744171	3·16	920099	1·40	824072	4·56	175928	18
43	744361	3·15	920015	1·40	824345	4·56	175655	17
44	744550	3·15	919931	1·41	824619	4·56	175381	16
45	744739	3·15	919846	1·41	824893	4·56	175107	15
46	744928	3·15	919762	1·41	825166	4·56	174834	14
47	745117	3·15	919677	1·41	825439	4·55	174561	13
48	745306	3·14	919593	1·41	825713	4·55	174287	12
49	745494	3·14	919508	1·41	825986	4·55	174014	11
50	745683	3·14	919424	1·41	826259	4·55	173741	10
51	9·745871	3·14	9·919339	1·41	9·826532	4·55	10·173468	9
52	746059	3·14	919254	1·41	826805	4·55	173195	8
53	746248	3·13	919169	1·41	827078	4·55	172922	7
54	746436	4·13	919085	1·41	827351	4·55	172649	6
55	746624	3·13	919000	1·41	827624	4·55	172376	5
56	746812	3·13	918915	1·42	827897	4·54	172103	4
57	746999	3·13	918830	1·42	828170	4·54	171830	3
58	747187	3·12	918745	1·42	828442	4·54	171558	2
59	747374	3·12	918659	1·42	828715	4·54	171285	1
60	747562	3·12	918574	1·42	828987	4·54	171013	0
	Cosine	D.	Sine	56°	Cotang.	D.	Tang.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9·747562	3·12	9·918574	1·42	9·828987	4·54	10·171013	60
1	747749	3·12	918489	1·42	829260	4·54	170740	59
2	747936	3·12	918404	1·42	829532	4·54	170468	58
3	748123	3·11	918318	1·42	829805	4·54	170195	57
4	748310	3·11	918233	1·42	830077	4·54	169923	56
5	748497	3·11	918147	1·42	830349	4·53	169651	55
6	748683	3·11	918062	1·42	830621	4·53	169379	54
7	748870	3·11	917976	1·43	830893	4·53	169107	53
8	749056	3·10	917891	1·43	831165	4·53	168835	52
9	749243	3·10	917805	1·43	831437	4·53	168563	51
10	749429	3·10	917719	1·43	831709	4·53	168291	50
11	9·749615	3·10	9·917634	1·43	9·831981	4·53	10·168019	49
12	749801	3·10	917548	1·43	832253	4·53	167747	48
13	749987	3·09	917462	1·43	832525	4·53	167475	47
14	750172	3·09	917376	1·43	832796	4·53	167204	46
15	750358	3·09	917290	1·43	833068	4·52	166932	45
16	750543	3·09	917204	1·43	833339	4·52	166661	44
17	750729	3·09	917118	1·44	833611	4·52	166389	43
18	750914	3·08	917032	1·44	833882	4·52	166118	42
19	751099	3·08	916946	1·44	834154	4·52	165846	41
20	751284	3·08	916859	1·44	834425	4·52	165575	40
21	9·751469	3·08	9·916773	1·44	9·834696	4·52	10·165304	39
22	751654	3·08	916687	1·44	834967	4·52	165033	38
23	751839	3·08	916600	1·44	835238	4·52	164762	37
24	752023	3·07	916514	1·44	835509	4·52	164491	36
25	752208	3·07	916427	1·44	835780	4·51	164220	35
26	752392	3·07	916341	1·44	836051	4·51	163949	34
27	752576	3·07	916254	1·44	836322	4·51	163678	33
28	752760	3·07	916167	1·45	836593	4·51	163407	32
29	752944	3·06	916081	1·45	836864	4·51	163136	31
30	753128	3·06	915994	1·45	837134	4·51	162866	30
31	9·753312	3·06	9·915907	1·45	9·837405	4·51	10·162595	29
32	753495	3·06	915820	1·45	837675	4·51	162325	28
33	753679	3·06	915733	1·45	837946	4·51	162054	27
34	753862	3·05	915646	1·45	838216	4·51	161784	26
35	754046	3·05	915559	1·45	838487	4·50	161513	25
36	754229	3·05	915472	1·45	838757	4·50	161243	24
37	754412	3·05	915385	1·45	839027	4·50	160973	23
38	754595	3·05	915297	1·45	839297	4·50	160703	22
39	754778	3·04	915210	1·45	839568	4·50	160432	21
40	754960	3·04	915123	1·46	839838	4·50	160162	20
41	9·755143	3·04	9·915035	1·46	9·840108	4·50	10·159892	19
42	755326	3·04	914948	1·46	840378	4·50	159622	18
43	755508	3·04	914860	1·46	840647	4·50	159353	17
44	755690	3·04	914773	1·46	840917	4·49	159083	16
45	755872	3·03	914685	1·46	841187	4·49	158813	15
46	756054	3·03	914598	1·46	841457	4·49	158543	14
47	756236	3·03	914510	1·46	841726	4·49	158274	13
48	756418	3·03	914422	1·46	841996	4·49	158004	12
49	756600	3·03	914334	1·46	842266	4·49	157734	11
50	756782	3·02	914246	1·47	842535	4·49	157465	10
51	9·756963	3·02	9·914158	1·47	9·842805	4·49	10·157195	9
52	757144	3·02	914070	1·47	843074	4·49	156926	8
53	757326	3·02	913982	1·47	843343	4·49	156657	7
54	757507	3·02	913894	1·47	843612	4·49	156388	6
55	757688	3·01	913806	1·47	843882	4·48	156118	5
56	757869	3·01	913718	1·47	844151	4·48	155849	4
57	758050	3·01	913630	1·47	844420	4·48	155580	3
58	758230	3·01	913541	1·47	844689	4·48	155311	2
59	758411	3·01	913453	1·47	844958	4·48	155042	1
60	758591	3·01	913365	1·47	845227	4·48	154773	0
	Cosine	D.	Sine	55°	Cotang.	D.	Tang.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9·758591	3·01	9·913365	1·47	9·845227	4·48	10·154773	60
1	758772	3·00	913276	1·47	845496	4·48	154504	59
2	758952	3·00	913187	1·48	845764	4·48	154236	58
3	759132	3·00	913099	1·48	846033	4·48	153967	57
4	759312	3·00	913010	1·48	846302	4·48	153698	56
5	759492	3·00	912922	1·48	846570	4·47	153430	55
6	759672	2·99	912833	1·48	846839	4·47	153161	54
7	759852	2·99	912744	1·48	847107	4·47	152893	53
8	760031	2·99	912655	1·48	847376	4·47	152624	52
9	760211	2·99	912566	1·48	847644	4·47	152356	51
10	760390	2·99	912477	1·48	847913	4·47	152087	50
11	9·760569	2·98	9·912388	1·48	9·848181	4·47	10·151819	49
12	760748	2·98	912299	1·49	848449	4·47	151551	48
13	760927	2·98	912210	1·49	848717	4·47	151283	47
14	761106	2·98	912121	1·49	848986	4·47	151014	46
15	761285	2·98	912031	1·49	849254	4·47	150746	45
16	761464	2·98	911942	1·49	849522	4·47	150478	44
17	761642	2·97	911853	1·49	849790	4·46	150210	43
18	761821	2·97	911763	1·49	850058	4·46	149942	42
19	761999	2·97	911674	1·49	850325	4·46	149675	41
20	762177	2·97	911584	1·49	850593	4·46	149407	40
21	9·762356	2·97	9·911495	1·49	9·850861	4·46	10·149139	39
22	762534	2·96	911405	1·49	851129	4·46	148871	38
23	762712	2·96	911315	1·50	851396	4·46	148604	37
24	762889	2·96	911226	1·50	851664	4·46	148336	36
25	763067	2·96	911136	1·50	851931	4·46	148069	35
26	763245	2·96	911046	1·50	852199	4·46	147801	34
27	763422	2·96	910956	1·50	852466	4·46	147534	33
28	763600	2·95	910866	1·50	852733	4·45	147267	32
29	763777	2·95	910776	1·50	853001	4·45	146999	31
30	763954	2·95	910686	1·50	853268	4·45	146732	30
31	9·764131	2·95	9·910596	1·50	9·853535	4·45	10·146465	29
32	764308	2·95	910506	1·50	853802	4·45	146198	28
33	764485	2·94	910415	1·50	854069	4·45	145931	27
34	764662	2·94	910325	1·51	854336	4·45	145664	26
35	764838	2·94	910235	1·51	854603	4·45	145397	25
36	765015	2·94	910144	1·51	854870	4·45	145130	24
37	765191	2·94	910054	1·51	855137	4·45	144863	23
38	765367	2·94	909963	1·51	855404	4·45	144596	22
39	765544	2·93	909873	1·51	855671	4·44	144329	21
40	765720	2·93	909782	1·51	855938	4·44	144062	20
41	9·765896	2·93	9·909691	1·51	9·856204	4·44	10·143796	19
42	766072	2·93	909601	1·51	856471	4·44	143529	18
43	766247	2·93	909510	1·51	856737	4·44	143263	17
44	766423	2·93	909419	1·51	857004	4·44	142996	16
45	766598	2·92	909328	1·52	857270	4·44	142730	15
46	766774	2·92	909237	1·52	857537	4·44	142463	14
47	766949	2·92	909146	1·52	857803	4·44	142197	13
48	767124	2·92	909055	1·52	858069	4·44	141931	12
49	767300	2·92	908964	1·52	858336	4·44	141664	11
50	767475	2·91	908873	1·52	858602	4·43	141398	10
51	9·767649	2·91	9·908781	1·52	9·858868	4·43	10·141132	9
52	767824	2·91	908690	1·52	859134	4·43	140866	8
53	767999	2·91	908599	1·52	859400	4·43	140600	7
54	768173	2·91	908507	1·52	859666	4·43	140334	6
55	768348	2·90	908416	1·53	859932	4·43	140068	5
56	768522	2·90	908324	1·53	860198	4·43	139802	4
57	768697	2·90	908233	1·53	860464	4·43	139536	3
58	768871	2·90	908141	1·53	860730	4·43	139270	2
59	769045	2·90	908049	1·53	860995	4·43	139005	1
60	769219	2·90	907958	1·53	861261	4·43	138739	0
	Cosine	D.	Sine	54°	Cotang	D.	Tang.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9·769219	2·90	9·907958	1·53	9·861261	4·43	10·138739	60
1	769393	2·89	907866	1·53	861527	4·43	138473	59
2	769566	2·89	907774	1·53	861792	4·42	138208	58
3	769740	2·89	907682	1·53	862058	4·42	137942	57
4	769913	2·89	907590	1·53	862323	4·42	137677	56
5	770087	2·89	907498	1·53	862589	4·42	137411	55
6	770260	2·88	907406	1·53	862854	4·42	137146	54
7	770433	2·88	907314	1·54	863119	4·42	136881	53
8	770606	2·88	907222	1·54	863385	4·42	136615	52
9	770779	2·88	907129	1·54	863650	4·42	136350	51
10	770952	2·88	907037	1·54	863915	4·42	136085	50
11	9·771125	2·88	9·906945	1·54	9·864180	4·42	10·135820	49
12	771298	2·87	906852	1·54	864445	4·42	135555	48
13	771470	2·87	906760	1·54	864710	4·42	135290	47
14	771643	2·87	906667	1·54	864975	4·41	135025	46
15	771815	2·87	906575	1·54	865240	4·41	134760	45
16	771987	2·87	906482	1·54	865505	4·41	134495	44
17	772159	2·87	906389	1·55	865770	4·41	134230	43
18	772331	2·86	906296	1·55	866035	4·41	133965	42
19	772503	2·86	906204	1·55	866300	4·41	133700	41
20	772675	2·86	906111	1·55	866564	4·41	133436	40
21	9·772847	2·86	9·906018	1·55	9·866829	4·41	10·133171	39
22	773018	2·86	905925	1·55	867094	4·41	132906	38
23	773190	2·86	905832	1·55	867358	4·41	132642	37
24	773361	2·85	905739	1·55	867623	4·41	132377	36
25	773533	2·85	905645	1·55	867887	4·41	132113	35
26	773704	2·85	905552	1·55	868152	4·40	131848	34
27	773875	2·85	905459	1·55	868416	4·40	131584	33
28	774046	2·85	905366	1·56	868680	4·40	131320	32
29	774217	2·85	905272	1·56	868945	4·40	131055	31
30	774388	2·84	905179	1·56	869209	4·40	130794	30
31	9·774558	2·84	9·905085	1·56	9·869473	4·40	10·130527	29
32	774729	2·84	904992	1·56	869737	4·40	130263	28
33	774899	2·84	904898	1·56	870001	4·40	129999	27
34	775070	2·84	904804	1·56	870265	4·40	129735	26
35	775240	2·84	904711	1·56	870529	4·40	129471	25
36	775410	2·83	904617	1·56	870793	4·40	129207	24
37	775580	2·83	904523	1·56	871057	4·40	128943	23
38	775750	2·83	904429	1·57	871321	4·40	128679	22
39	775920	2·83	904335	1·57	871585	4·40	128415	21
40	776090	2·83	904241	1·57	871849	4·39	128151	20
41	9·776259	2·83	9·904147	1·57	9·872112	4·39	10·127888	19
42	776429	2·82	904053	1·57	872376	4·39	127624	18
43	776598	2·82	903959	1·57	872640	4·39	127360	17
44	776768	2·82	903864	1·57	872903	4·39	127097	16
45	776937	2·82	903770	1·57	873167	4·39	126833	15
46	777106	2·82	903676	1·57	873430	4·39	126570	14
47	777275	2·81	903581	1·57	873694	4·39	126306	13
48	777444	2·81	903487	1·57	873957	4·39	126043	12
49	777613	2·81	903392	1·58	874220	4·39	125780	11
50	777781	2·81	903298	1·58	874484	4·39	125516	10
51	9·777950	2·81	9·903203	1·58	9·874747	4·39	10·125253	9
52	778119	2·81	903108	1·58	875010	4·39	124990	8
53	778287	2·80	903014	1·58	875273	4·38	124727	7
54	778455	2·80	902919	1·58	875536	4·38	124464	6
55	778624	2·80	902824	1·58	875800	4·38	124200	5
56	778792	2·80	902729	1·58	876063	4·38	123937	4
57	778960	2·80	902634	1·58	876326	4·38	123674	3
58	779128	2·80	902539	1·59	876589	4·38	123411	2
59	779295	2·79	902444	1·59	876851	4·38	123149	1
60	779463	2·79	902349	1·59	877114	4·38	122886	0
	Cosine	D.	Sine	53°	Cotang.	D.	Tang.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9·779463	2·79	9·902349	1·59	9·877114	4·38	10·122886	60
1	779631	2·79	902253	1·59	877377	4·38	122623	59
2	779798	2·79	902158	1·59	877640	4·38	122360	58
3	779966	2·79	902063	1·59	877903	4·38	122097	57
4	780133	2·79	901967	1·59	878165	4·38	121835	56
5	780300	2·78	901872	1·59	878428	4·38	121572	55
6	780467	2·78	901776	1·59	878691	4·38	121309	54
7	780634	2·78	901681	1·59	878953	4·37	121047	53
8	780801	2·78	901585	1·59	879216	4·37	120784	52
9	780968	2·78	901490	1·59	879478	4·37	120522	51
10	781134	2·78	901394	1·60	879741	4·37	120259	50
11	9·781301	2·77	9·901298	1·60	9·880003	4·37	10·119997	49
12	781468	2·77	901202	1·60	880265	4·37	119735	48
13	781634	2·77	901106	1·60	880528	4·37	119472	47
14	781800	2·77	901010	1·60	880790	4·37	119210	46
15	781966	2·77	900914	1·60	881052	4·37	118948	45
16	782132	2·77	900818	1·60	881314	4·37	118686	44
17	782298	2·76	900722	1·60	881576	4·37	118424	43
18	782464	2·76	900626	1·60	881839	4·37	118161	42
19	782630	2·76	900529	1·60	882101	4·37	117899	41
20	782796	2·76	900433	1·61	882363	4·36	117637	40
21	9·782961	2·76	9·900337	1·61	9·882625	4·36	10·117375	39
22	783127	2·76	900240	1·61	882887	4·36	117113	38
23	783292	2·75	900144	1·61	883148	4·36	116852	37
24	783458	2·75	900047	1·61	883410	4·36	116590	36
25	783623	2·75	899951	1·61	883672	4·36	116328	35
26	783788	2·75	899854	1·61	883934	4·36	116066	34
27	783953	2·75	899757	1·61	884196	4·36	115804	33
28	784118	2·75	899660	1·61	884457	4·36	115543	32
29	784282	2·74	899564	1·61	884719	4·36	115281	31
30	784447	2·74	899467	1·62	884980	4·36	115020	30
31	9·784612	2·74	9·899370	1·62	9·885242	4·36	10·114758	29
32	784776	2·74	899273	1·62	885503	4·36	114497	28
33	784941	2·74	899176	1·62	885765	4·36	114235	27
34	785105	2·74	899078	1·62	886026	4·36	113974	26
35	785269	2·73	898981	1·62	886288	4·36	113712	25
36	785433	2·73	898884	1·62	886549	4·35	113451	24
37	785597	2·73	898787	1·62	886810	4·35	113190	23
38	785761	2·73	898689	1·62	887072	4·35	112928	22
39	785925	2·73	898592	1·62	887333	4·35	112667	21
40	786089	2·73	898494	1·63	887594	4·35	112406	20
41	9·786252	2·72	9·898397	1·63	9·887855	4·35	10·112145	19
42	786416	2·72	898299	1·63	888116	4·35	111884	18
43	786579	2·72	898202	1·63	888377	4·35	111623	17
44	786742	2·72	898104	1·63	888639	4·35	111361	16
45	786906	2·72	898006	1·63	888900	4·35	111100	15
46	787069	2·72	897908	1·63	889160	4·35	110840	14
47	787232	2·71	897810	1·63	889421	4·35	110579	13
48	787395	2·71	897712	1·63	889682	4·35	110318	12
49	787557	2·71	897614	1·63	889943	4·35	110057	11
50	787720	2·71	897516	1·63	890204	4·34	109796	10
51	9·787883	2·71	9·897418	1·64	9·890465	4·34	10·109535	9
52	788045	2·71	897320	1·64	890725	4·34	109275	8
53	788208	2·71	897222	1·64	890986	4·34	109014	7
54	788370	2·70	897123	1·64	891247	4·34	108753	6
55	788532	2·70	897025	1·64	891507	4·34	108493	5
56	788694	2·70	896926	1·64	891768	4·34	108232	4
57	788856	2·70	896828	1·64	892028	4·34	107972	3
58	789018	2·70	896729	1·64	892289	4·34	107711	2
59	789180	2·70	896631	1·64	892549	4·34	107451	1
60	789342	2·69	896532	1·64	892810	4·34	107190	0
	Cosine	D.	Sine	52°	Cotang.	D.	Tang.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9.789342	2.69	9.896532	1.64	9.892810	4.34	10.107190	60
1	789504	2.69	896433	1.65	893070	4.34	106930	59
2	789665	2.69	896335	1.65	893331	4.34	106669	58
3	789827	2.69	896236	1.65	893591	4.34	106409	57
4	789988	2.69	896137	1.65	893851	4.34	106149	56
5	790149	2.69	896038	1.65	894111	4.34	105889	55
6	790310	2.68	895939	1.65	894371	4.34	105629	54
7	790471	2.68	895840	1.65	894632	4.33	105368	53
8	790632	2.68	895741	1.65	894892	4.33	105108	52
9	790793	2.68	895641	1.65	895152	4.33	104848	51
10	790954	2.68	895542	1.65	895412	4.33	104588	50
11	9.791115	2.68	9.895443	1.66	9.895672	4.33	10.104328	49
12	791275	2.67	895343	1.66	895932	4.33	104068	48
13	791436	2.67	895244	1.66	896192	4.33	103808	47
14	791596	2.67	895145	1.66	896452	4.33	103548	46
15	791757	2.67	895045	1.66	896712	4.33	103288	45
16	791917	2.67	894945	1.66	896971	4.33	103029	44
17	792077	2.67	894846	1.66	897231	4.33	102769	43
18	792237	2.66	894746	1.66	897491	4.33	102509	42
19	792397	2.66	894646	1.66	897751	4.33	102249	41
20	792557	2.66	894546	1.66	898010	4.33	101990	40
21	9.792716	2.66	9.894446	1.67	9.898270	4.33	10.101730	39
22	792876	2.66	894346	1.67	898530	4.33	101470	38
23	793035	2.66	894246	1.67	898789	4.33	101211	37
24	793195	2.65	894146	1.67	899049	4.32	100951	36
25	793354	2.65	894046	1.67	899308	4.32	100692	35
26	793514	2.65	893946	1.67	899568	4.32	100432	34
27	793673	2.65	893846	1.67	899827	4.32	100173	33
28	793832	2.65	893745	1.67	900086	4.32	099914	32
29	793991	2.65	893645	1.67	900346	4.32	099654	31
30	794150	2.64	893544	1.67	900605	4.32	099395	30
31	9.794308	2.64	9.893444	1.68	9.900864	4.32	10.099136	29
32	794467	2.64	893343	1.68	901124	4.32	098876	28
33	794626	2.64	893243	1.68	901383	4.32	098617	27
34	794784	2.64	893142	1.68	901642	4.32	098358	26
35	794942	2.64	893041	1.68	901901	4.32	098099	25
36	795101	2.64	892940	1.68	902160	4.32	097840	24
37	795259	2.63	892839	1.68	902419	4.32	097581	23
38	795417	2.63	892739	1.68	902679	4.32	097321	22
39	795575	2.63	892638	1.68	902938	4.32	097062	21
40	795733	2.63	892536	1.68	903197	4.31	096803	20
41	9.795891	2.63	9.892435	1.69	9.903455	4.31	10.096545	19
42	796049	2.63	892334	1.69	903714	4.31	096286	18
43	796206	2.63	892233	1.69	903973	4.31	096027	17
44	796364	2.62	892132	1.69	904232	4.31	095768	16
45	796521	2.62	892030	1.69	904491	4.31	095509	15
46	796679	2.62	891929	1.69	904750	4.31	095250	14
47	796836	2.62	891827	1.69	905008	4.31	094992	13
48	796993	2.62	891726	1.69	905267	4.31	094733	12
49	797150	2.61	891624	1.69	905526	4.31	094474	11
50	797307	2.61	891523	1.70	905784	4.31	094216	10
51	9.797464	2.61	9.891421	1.70	9.906043	4.31	10.093957	9
52	797621	2.61	891319	1.70	906302	4.31	093698	8
53	797777	2.61	891217	1.70	906560	4.31	093440	7
54	797934	2.61	891115	1.70	906819	4.31	093181	6
55	798091	2.61	891013	1.70	907077	4.31	092923	5
56	798247	2.61	890911	1.70	907336	4.31	092664	4
57	798403	2.60	890809	1.70	907594	4.31	092406	3
58	798560	2.60	890707	1.70	907852	4.31	092148	2
59	798716	2.60	890605	1.70	908111	4.30	091889	1
60	798872	2.60	890503	1.70	908369	4.30	091631	0
	Cosine	D.	Sine	51°	Cotang.	D.	Tang.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9·798872	2·60	9·890503	1·70	9·908369	4·30	10·091631	60
1	799028	2·60	890400	1·71	908628	4·30	091372	59
2	799184	2·60	890298	1·71	908886	4·30	091114	58
3	799339	2·59	890195	1·71	909144	4·30	090856	57
4	799495	2·59	890093	1·71	909402	4·30	090598	56
5	799651	2·59	889990	1·71	909660	4·30	090340	55
6	799806	2·59	889888	1·71	909918	4·30	090082	54
7	799962	2·59	889785	1·71	910177	4·30	089823	53
8	800117	2·59	889682	1·71	910435	4·30	089565	52
9	800272	2·58	889579	1·71	910693	4·30	089307	51
10	800427	2·58	889477	1·71	910951	4·30	089049	50
11	9·800582	2·58	9·889374	1·72	9·911209	4·30	10·088791	49
12	800737	2·58	889271	1·72	911467	4·30	088533	48
13	800892	2·58	889168	1·72	911724	4·30	088276	47
14	801047	2·58	889064	1·72	911982	4·30	088018	46
15	801201	2·58	888961	1·72	912240	4·30	087760	45
16	801356	2·57	888858	1·72	912498	4·30	087502	44
17	801511	2·57	888755	1·72	912756	4·30	087244	43
18	801665	2·57	888651	1·72	913014	4·29	086986	42
19	801819	2·57	888548	1·72	913271	4·29	086729	41
20	801973	2·57	888444	1·73	913529	4·29	086471	40
21	9·802128	2·57	9·888341	1·73	9·913787	4·29	10·086213	39
22	802282	2·56	888237	1·73	914044	4·29	085956	38
23	802436	2·56	888134	1·73	914302	4·29	085698	37
24	802589	2·56	888030	1·73	914560	4·29	085440	36
25	802743	2·56	887926	1·73	914817	4·29	085183	35
26	802897	2·56	887822	1·73	915075	4·29	084925	34
27	803050	2·56	887718	1·73	915332	4·29	084668	33
28	803204	2·56	887614	1·73	915590	4·29	084410	32
29	803357	2·55	887510	1·73	915847	4·29	084153	31
30	803511	2·55	887406	1·74	916104	4·29	083896	30
31	9·803664	2·55	9·887302	1·74	9·916362	4·29	10·083638	29
32	803817	2·55	887198	1·74	916619	4·29	083381	28
33	803970	2·55	887093	1·74	916877	4·29	083123	27
34	804123	2·55	886989	1·74	917134	4·29	082866	26
35	804276	2·54	886885	1·74	917391	4·29	082609	25
36	804428	2·54	886780	1·74	917648	4·29	082352	24
37	804581	2·54	886676	1·74	917905	4·29	082095	23
38	804734	2·54	886571	1·74	918163	4·28	081837	22
39	804886	2·54	886466	1·74	918420	4·28	081580	21
40	805039	2·54	886362	1·75	918677	4·28	081323	20
41	9·805191	2·54	9·886257	1·75	9·918934	4·28	10·081066	19
42	805343	2·53	886152	1·75	919191	4·28	080809	18
43	805495	2·53	886047	1·75	919448	4·28	080552	17
44	805647	2·53	885942	1·75	919705	4·28	080295	16
45	805799	2·53	885837	1·75	919962	4·28	080038	15
46	805951	2·53	885732	1·75	920219	4·28	079781	14
47	806103	2·53	885627	1·75	920476	4·28	079524	13
48	806254	2·53	885522	1·75	920733	4·28	079267	12
49	806406	2·52	885416	1·75	920990	4·28	079010	11
50	806557	2·52	885311	1·76	921247	4·28	078753	10
51	9·806709	2·52	9·885205	1·76	9·921503	4·28	10·078497	9
52	806860	2·52	885100	1·76	921760	4·28	078240	8
53	807011	2·52	884994	1·76	922017	4·28	077983	7
54	807163	2·52	884889	1·76	922274	4·28	077726	6
55	807314	2·52	884783	1·76	922530	4·28	077470	5
56	807465	2·51	884677	1·76	922787	4·28	077213	4
57	807615	2·51	884572	1·76	923044	4·28	076956	3
58	807766	2·51	884466	1·76	923300	4·28	076700	2
59	807917	2·51	884360	1·76	923557	4·27	076443	1
60	808067	2·51	884254	1·77	923813	4·27	076187	0
	Cosine	D.	Sine	50°	Cotang.	D.	Tang.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9·808067	2·51	9·884254	1·77	9·923813	4·27	10·076187	60
1	808218	2·51	884148	1·77	924070	4·27	075930	59
2	808368	2·51	884042	1·77	924327	4·27	075673	58
3	808519	2·50	883936	1·77	924583	4·27	075417	57
4	808669	2·50	883829	1·77	924840	4·27	075160	56
5	808819	2·50	883723	1·77	925096	4·27	074904	55
6	808969	2·50	883617	1·77	925352	4·27	074648	54
7	809119	2·50	883510	1·77	925609	4·27	074391	53
8	809269	2·50	883404	1·77	925865	4·27	074135	52
9	809419	2·49	883297	1·78	926122	4·27	073878	51
10	809569	2·49	883191	1·78	926378	4·27	073622	50
11	9·809718	2·49	9·883084	1·78	9·926634	4·27	10·073366	49
12	809868	2·49	882977	1·78	926890	4·27	073110	48
13	810017	2·49	882871	1·78	927147	4·27	072853	47
14	810167	2·49	882764	1·78	927403	4·27	072597	46
15	810316	2·48	882657	1·78	927659	4·27	072341	45
16	810465	2·48	882550	1·78	927915	4·27	072085	44
17	810614	2·48	882443	1·78	928171	4·27	071829	43
18	810763	2·48	882336	1·79	928427	4·27	071573	42
19	810912	2·48	882229	1·79	928683	4·27	071317	41
20	811061	2·48	882121	1·79	928940	4·27	071060	40
21	9·811210	2·48	9·882014	1·79	9·929196	4·27	10·070804	39
22	811358	2·47	881907	1·79	929452	4·27	070548	38
23	811507	2·47	881799	1·79	929708	4·27	070292	37
24	811655	2·47	881692	1·79	929964	4·26	070036	36
25	811804	2·47	881584	1·79	930220	4·26	069780	35
26	811952	2·47	881477	1·79	930475	4·26	069525	34
27	812100	2·47	881369	1·79	930731	4·26	069269	33
28	812248	2·47	881261	1·80	930987	4·26	069013	32
29	812396	2·46	881153	1·80	931243	4·26	068757	31
30	812544	2·46	881046	1·80	931499	4·26	068501	30
31	9·812692	2·46	9·880938	1·80	9·931755	4·26	10·068245	29
32	812840	2·46	880830	1·80	•932010	4·26	067990	28
33	812988	2·46	880722	1·80	932266	4·26	067734	27
34	813135	2·46	880613	1·80	932522	4·26	067478	26
35	813283	2·46	880505	1·80	932778	4·26	067222	25
36	813430	2·45	880397	1·80	933033	4·26	066967	24
37	813578	2·45	880289	1·81	933289	4·26	066711	23
38	813725	2·45	880180	1·81	933545	4·26	066455	22
39	813872	2·45	880072	1·81	933800	4·26	066200	21
40	814019	2·45	879963	1·81	934056	4·26	065944	20
41	9·814166	2·45	9·879855	1·81	9·934311	4·26	10·065689	19
42	814313	2·45	879746	1·81	934567	4·26	065433	18
43	814460	2·44	879637	1·81	934823	4·26	065177	17
44	814607	2·44	879529	1·81	935078	4·26	064922	16
45	814753	2·44	879420	1·81	935333	4·26	064667	15
46	814900	2·44	879311	1·81	935589	4·26	064411	14
47	815046	2·44	879202	1·82	935844	4·26	064156	13
48	815193	2·44	879093	1·82	936100	4·26	063900	12
49	815339	2·44	878984	1·82	936355	4·26	063645	11
50	815485	2·43	878875	1·82	936610	4·26	063390	10
51	9·815631	2·43	9·878766	1·82	9·936866	4·25	10·063134	9
52	815778	2·43	878656	1·82	937121	4·25	062879	8
53	815924	2·43	878547	1·82	937376	4·25	062624	7
54	816069	2·43	878438	1·82	937632	4·25	062368	6
55	816215	2·43	878328	1·82	937887	4·25	062113	5
56	816361	2·43	878219	1·83	938142	4·25	061858	4
57	816507	2·42	878109	1·83	938398	4·25	061602	3
58	816652	2·42	877999	1·83	938653	4·25	061347	2
59	816798	2·42	877890	1·83	938908	4·25	061092	1
60	816943	2·42	877780	1·83	939163	4·25	060837	0
	Cosine	D.	Sine	49°	Cotang.	D.	Tang.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9·816943	2·42	9·877780	1·83	9·939163	4·25	10·060837	60
1	817088	2·42	877670	1·83	939418	4·25	060582	59
2	817233	2·42	877560	1·83	939673	4·25	060327	58
3	817379	2·42	877450	1·83	939928	4·25	060072	57
4	817524	2·41	877340	1·83	940183	4·25	059817	56
5	817668	2·41	877230	1·84	940438	4·25	059562	55
6	817813	2·41	877120	1·84	940694	4·25	059306	54
7	817958	2·41	877010	1·84	940949	4·25	059051	53
8	818103	2·41	876899	1·84	941204	4·25	058796	52
9	818247	2·41	876789	1·84	941458	4·25	058542	51
10	818392	2·41	876678	1·84	941714	4·25	058286	50
11	9·818536	2·40	9·876568	1·84	9·941968	4·25	10·058032	49
12	818681	2·40	876457	1·84	942223	4·25	057777	48
13	818825	2·40	876347	1·84	942478	4·25	057522	47
14	818969	2·40	876236	1·85	942733	4·25	057267	46
15	819113	2·40	876125	1·85	942988	4·25	057012	45
16	819257	2·40	876014	1·85	943243	4·25	056757	44
17	819401	2·40	875904	1·85	943498	4·25	056502	43
18	819545	2·39	875793	1·85	943752	4·25	056248	42
19	819689	2·39	875682	1·85	944007	4·25	055993	41
20	819832	2·39	875571	1·85	944262	4·25	055738	40
21	9·819976	2·39	9·875459	1·85	9·944517	4·25	10·055483	39
22	820120	2·39	875348	1·85	944771	4·24	055229	38
23	820263	2·39	875237	1·85	945026	4·24	054974	37
24	820406	2·39	875126	1·86	945281	4·24	054719	36
25	820550	2·38	875014	1·86	945535	4·24	054465	35
26	820693	2·38	874903	1·86	945790	4·24	054210	34
27	820836	2·38	874791	1·86	946045	4·24	053955	33
28	820979	2·38	874680	1·86	946299	4·24	053701	32
29	821122	2·38	874568	1·86	946554	4·24	053446	31
30	821265	2·38	874456	1·86	946808	4·24	053192	30
31	9·821407	2·38	9·874344	1·86	9·947063	4·24	10·052937	29
32	821550	2·38	874232	1·87	947318	4·24	052682	28
33	821693	2·37	874121	1·87	947572	4·24	052428	27
34	821835	2·37	874009	1·87	947826	4·24	052174	26
35	821977	2·37	873896	1·87	948081	4·24	051919	25
36	822120	2·37	873784	1·87	948336	4·24	051664	24
37	822262	2·37	873672	1·87	948590	4·24	051410	23
38	822404	2·37	873560	1·87	948844	4·24	051156	22
39	822546	2·37	873448	1·87	949099	4·24	050901	21
40	822688	2·36	873335	1·87	949353	4·24	050647	20
41	9·822830	2·36	9·873223	1·87	9·949607	4·24	10·050393	19
42	822972	2·36	873110	1·88	949862	4·24	050138	18
43	823114	2·36	872998	1·88	950116	4·24	049884	17
44	823255	2·36	872885	1·88	950370	4·24	049630	16
45	823397	2·36	872772	1·88	950625	4·24	049375	15
46	823539	2·36	872659	1·88	950879	4·24	049121	14
47	823680	2·35	872547	1·88	951133	4·24	048867	13
48	823821	2·35	872434	1·88	951388	4·24	048612	12
49	823963	2·35	872321	1·88	951642	4·24	048358	11
50	824104	2·35	872208	1·88	951896	4·24	048104	10
51	9·824245	2·35	9·872095	1·89	9·952150	4·24	10·047850	9
52	824386	2·35	871981	1·89	952405	4·24	047595	8
53	824527	2·35	871868	1·89	952659	4·24	047341	7
54	824668	2·34	871755	1·89	952913	4·24	047087	6
55	824808	2·34	871641	1·89	953167	4·23	046833	5
56	824949	2·34	871528	1·89	953421	4·23	046579	4
57	825090	2·34	871414	1·89	953675	4·23	046325	3
58	825230	2·34	871301	1·89	953929	4·23	046071	2
59	825371	2·34	871187	1·89	954183	4·23	045817	1
60	825511	2·34	871073	1·90	954437	4·23	045563	0
	Cosine	D.	Sine	48°	Cotang.	D.	Tang.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9·825511	2·34	9·871073	1·90	9·954437	4·23	10·045563	60
1	825651	2·33	870960	1·90	954691	4·23	045309	59
2	825791	2·33	870846	1·90	954945	4·23	045055	58
3	825931	2·33	870732	1·90	955200	4·23	044800	57
4	826071	2·33	870618	1·90	955454	4·23	044546	56
5	826211	2·33	870504	1·90	955707	4·23	044293	55
6	826351	2·33	870390	1·90	955961	4·23	044039	54
7	826491	2·33	870276	1·90	956215	4·23	043785	53
8	826631	2·33	870161	1·90	956469	4·23	043531	52
9	826770	2·32	870047	1·91	956723	4·23	043277	51
10	826910	2·32	869933	1·91	956977	4·23	043023	50
11	9·827049	2·32	9·869818	1·91	9·957231	4·23	10·042769	49
12	827189	2·32	869704	1·91	957485	4·23	042515	48
13	827328	2·32	869589	1·91	957739	4·23	042261	47
14	827467	2·32	869474	1·91	957993	4·23	042007	46
15	827606	2·32	869360	1·91	958246	4·23	041754	45
16	827745	2·32	869245	1·91	958500	4·23	041500	44
17	827884	2·31	869130	1·91	958754	4·23	041246	43
18	828023	2·31	869015	1·92	959008	4·23	040992	42
19	828162	2·31	868900	1·92	959262	4·23	040738	41
20	828301	2·31	868785	1·92	959516	4·23	040484	40
21	9·828439	2·31	9·868670	1·92	9·959769	4·23	10·040231	39
22	828578	2·31	868555	1·92	960023	4·23	039977	38
23	828716	2·31	868440	1·92	960277	4·23	039723	37
24	828855	2·30	868324	1·92	960531	4·23	039469	36
25	828993	2·30	868209	1·92	960784	4·23	039216	35
26	829131	2·30	868093	1·92	961038	4·23	038962	34
27	829269	2·30	867978	1·93	961291	4·23	038709	33
28	829407	2·30	867862	1·93	961545	4·23	038455	32
29	829545	2·30	867747	1·93	961799	4·23	038201	31
30	829683	2·30	867631	1·93	962052	4·23	037948	30
31	9·829821	2·29	9·867515	1·93	9·962306	4·23	10·037694	29
32	829959	2·29	867399	1·93	962560	4·23	037440	28
33	830097	2·29	867283	1·93	962813	4·23	037187	27
34	830234	2·29	867167	1·93	963067	4·23	036933	26
35	830372	2·29	867051	1·93	963320	4·23	036680	25
36	830509	2·29	866935	1·94	963574	4·23	036426	24
37	830646	2·29	866819	1·94	963827	4·23	036173	23
38	830784	2·29	866703	1·94	964081	4·23	035919	22
39	830921	2·28	866586	1·94	964335	4·23	035665	21
40	831058	2·28	866470	1·94	964588	4·22	035412	20
41	9·831195	2·28	9·866353	1·94	9·964842	4·22	10·035158	19
42	831332	2·28	866237	1·94	965095	4·22	034905	18
43	831469	2·28	866120	1·94	965349	4·22	034651	17
44	831606	2·28	866004	1·95	965602	4·22	034398	16
45	831742	2·28	865887	1·95	965855	4·22	034145	15
46	831879	2·28	865770	1·95	966105	4·22	033891	14
47	832015	2·27	865653	1·95	966362	4·22	033638	13
48	832152	2·27	865536	1·95	966616	4·22	033384	12
49	832288	2·27	865419	1·95	966869	4·22.	033131	11
50	832425	2·27	865302	1·95	967123	4·22	032877	10
51	9·832561	2·27	9·865185	1·95	9·967376	4·22	10·032624	9
52	832697	2·27	865068	1·95	967629	4·22	032371	8
53	832833	2·27	864950	1·95	967883	4·22	032117	7
54	832969	2·26	864833	1·96	968136	4·22	031864	6
55	833105	2·26	864716	1·96	968389	4·22	031611	5
56	833241	2·26	864598	1·96	968643	4·22	031357	4
57	833377	2·26	864481	1·96	968896	4·22	031104	3
58	833512	2·26	864363	1·96	969149	4·22	030851	2
59	833648	2·26	864245	1·96	969403	4·22	030597	1
60	833783	2·26	864127	1·96	969656	4·22	030344	0
	Cosine	D.	Sine	47°	Cotang.	D.	Tang.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9·833783	2·26	9·864127	1·96	9·969656	4·22	10·030344	60
1	833919	2·25	864010	1·96	969909	4·22	030091	59
2	834054	2·25	863892	1·97	970162	4·22	029838	58
3	834189	2·25	863774	1·97	970416	4·22	029584	57
4	834325	2·25	863656	1·97	970669	4·22	029331	56
5	834460	2·25	863538	1·97	970922	4·22	029078	55
6	834595	2·25	863419	1·97	971175	4·22	028825	54
7	834730	2·25	863301	1·97	971429	4·22	028571	53
8	834865	2·25	863183	1·97	971682	4·22	028318	52
9	834999	2·24	863064	1·97	971935	4·22	028065	51
10	835134	2·24	862946	1·98	972188	4·22	027812	50
11	9·835269	2·24	9·862827	1·98	9·972441	4·21	10·027559	49
12	835403	2·24	862709	1·98	972694	4·22	027306	48
13	835538	2·24	862590	1·98	972948	4·22	027052	47
14	835672	2·24	862471	1·98	973201	4·22	026799	46
15	835807	2·24	862353	1·98	973454	4·22	026546	45
16	835941	2·24	862234	1·98	973707	4·22	026293	44
17	836075	2·23	862115	1·98	973960	4·22	026040	43
18	836209	2·23	861996	1·98	974213	4·22	025787	42
19	836343	2·23	861877	1·98	974466	4·22	025534	41
20	836477	2·23	861758	1·99	974719	4·22	025281	40
21	9·836611	2·23	9·861638	1·99	9·974973	4·22	10·025027	39
22	836745	2·23	861519	1·99	975226	4·22	024774	38
23	836878	2·23	861400	1·99	975479	4·22	024521	37
24	837012	2·22	861280	1·99	975732	4·22	024268	36
25	837146	2·22	861161	1·99	975985	4·22	024015	35
26	837279	2·22	861041	1·99	976238	4·22	023762	34
27	837412	2·22	860922	1·99	976491	4·22	023509	33
28	837546	2·22	860802	1·99	976744	4·22	023256	32
29	837679	2·22	860682	2·00	976997	4·22	023003	31
30	837812	2·22	860562	2·00	977250	4·22	022750	30
31	9·837945	2·22	9·860442	2·00	9·977503	4·22	10·022497	29
32	838078	2·21	860322	2·00	977756	4·22	022244	28
33	838211	2·21	860202	2·00	978009	4·22	021991	27
34	838344	2·21	860082	2·00	978262	4·22	021738	26
35	838477	2·21	859962	2·00	978515	4·22	021485	25
36	838610	2·21	859842	2·00	978768	4·22	021232	24
37	838742	2·21	859721	2·01	979021	4·22	020979	23
38	838875	2·21	859601	2·01	979274	4·22	020726	22
39	839007	2·21	859480	2·01	979527	4·22	020473	21
40	839140	2·20	859360	2·01	979780	4·22	020220	20
41	9·839272	2·20	9·859239	2·01	9·980033	4·22	10·019967	19
42	839404	2·20	859119	2·01	980286	4·22	019714	18
43	839536	2·20	858998	2·01	980538	4·22	019462	17
44	839668	2·20	858877	2·01	980791	4·21	019209	16
45	839800	2·20	858756	2·02	981044	4·21	018956	15
46	839932	2·20	858635	2·02	981297	4·21	018703	14
47	840064	2·19	858514	2·02	981550	4·21	018450	13
48	840196	2·19	858393	2·02	981803	4·21	018197	12
49	840328	2·19	858272	2·02	982056	4·21	017944	11
50	840459	2·19	858151	2·02	982309	4·21	017691	10
51	9·840591	2·19	9·858029	2·02	9·982562	4·21	10·017438	9
52	840722	2·19	857908	2·02	982814	4·21	017186	8
53	840854	2·19	857786	2·02	983067	4·21	016933	7
54	840985	2·19	857665	2·03	983320	4·21	016680	6
55	841116	2·18	857543	2·03	983573	4·21	016427	5
56	841247	2·18	857422	2·03	983826	4·21	016174	4
57	841378	2·18	857300	2·03	984079	4·21	015921	3
58	841509	2·18	857178	2·03	984331	4·21	015669	2
59	841640	2·18	857056	2·03	984584	4·21	015416	1
60	841771	2·18	856934	2·03	984837	4·21	015163	0
	Cosine	D.	Sine	46°	Cotang.	D.	Tang.	M.

M.	Sine	D.	Cosine	D.	Tang.	D.	Cotang.	
0	9·841771	2·18	9·856934	2·03	9·984837	4·21	10·015163	60
1	841902	2·18	856812	2·03	985090	4·21	014910	59
2	842033	2·18	856690	2·04	985343	4·21	014657	58
3	842163	2·17	856568	2·04	985596	4·21	014404	57
4	842294	2·17	856446	2·04	985848	4·21	014152	56
5	842424	2·17	856323	2·04	986101	4·21	013899	55
6	842555	2·17	856201	2·04	986354	4·21	013646	54
7	842685	2·17	856078	2·04	986607	4·21	013393	53
8	842815	2·17	855956	2·04	986860	4·21	013140	52
9	842946	2·17	855833	2·04	987112	4·21	012888	51
10	843076	2·17	855711	2·05	987365	4·21	012635	50
11	9·843206	2·16	9·855588	2·05	9·987618	4·21	10·012382	49
12	843336	2·16	855465	2·05	987871	4·21	012129	48
13	843466	2·16	855342	2·05	988123	4·21	011877	47
14	843595	2·16	855219	2·05	988376	4·21	011624	46
15	843725	2·16	855096	2·05	988629	4·21	011371	45
16	843855	2·16	854973	2·05	988882	4·21	011118	44
17	843984	2·16	854850	2·05	989134	4·21	010866	43
18	844114	2·15	854727	2·06	989387	4·21	010613	42
19	844243	2·15	854603	2·06	989640	4·21	010360	41
20	844372	2·15	854480	2·06	989893	4·21	010107	40
21	9·844502	2·15	9·854356	2·06	9·990145	4·21	10·009855	39
22	844631	2·15	854233	2·06	990398	4·21	009602	38
23	844760	2·15	854109	2·06	990651	4·21	009349	37
24	844889	2·15	853986	2·06	990903	4·21	009097	36
25	845018	2·15	853862	2·06	991156	4·21	008844	35
26	845147	2·15	853738	2·06	991409	4·21	008591	34
27	845276	2·14	853614	2·07	991662	4·21	008338	33
28	845405	2·14	853490	2·07	991914	4·21	008086	32
29	845533	2·14	853366	2·07	992167	4·21	007833	31
30	845662	2·14	853242	2·07	992420	4·21	007580	30
31	9·845790	2·14	9·853118	2·07	9·992672	4·21	10·007328	29
32	845919	2·14	852994	2·07	992925	4·21	007075	28
33	846047	2·14	852869	2·07	993178	4·21	006822	27
34	846175	2·14	852745	2·07	993430	4·21	006570	26
35	846304	2·14	852620	2·07	993683	4·21	006317	25
36	846432	2·13	852496	2·08	993936	4·21	006064	24
37	846560	2·13	852371	2·08	994189	4·21	005811	23
38	846688	2·13	852247	2·08	994441	4·21	005559	22
39	846816	2·13	852122	2·08	994694	4·21	005306	21
40	846944	2·13	851997	2·08	994947	4·21	005053	20
41	9·847071	2·13	9·851872	2·08	9·995199	4·21	10·004801	19
42	847199	2·13	851747	2·08	995452	4·21	004548	18
43	847327	2·13	851622	2·08	995705	4·21	004295	17
44	847454	2·12	851497	2·09	995957	4·21	004043	16
45	847582	2·12	851372	2·09	996210	4·21	003790	15
46	847709	2·12	851246	2·09	996463	4·21	003537	14
47	847836	2·12	851121	2·09	996715	4·21	003285	13
48	847964	2·12	850996	2·09	996968	4·21	003032	12
49	848091	2·12	850870	2·09	997221	4·21	002779	11
50	848218	2·12	850745	2·09	997473	4·21	002527	10
51	9·848345	2·12	9·850619	2·09	9·997726	4·21	10·002274	9
52	848472	2·11	850493	2·10	997979	4·21	002021	8
53	848599	2·11	850368	2·10	998231	4·21	001769	7
54	848726	2·11	850242	2·10	998484	4·21	001516	6
55	848852	2·11	850116	2·10	998737	4·21	001263	5
56	848979	2·11	849990	2·10	998989	4·21	001011	4
57	849106	2·11	849864	2·10	999242	4·21	000758	3
58	849232	2·11	849738	2·10	999495	4·21	000505	2
59	849359	2·11	849611	2·10	999748	4·21	000253	1
60	849485	2·11	849485	2·10	10·000000	4·21	10·000000	0
	Cosine	D.	Sine	45°	Cotang.	D.	Tang.	M.

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